Computer Algebra and Computer Science

It's complicated...

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June 21st, 2018

ACA'18 – Santiago de Compostela

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Context: SC^2

EU H2020-FETOPEN-2015-CSA 712689

Satisfiability Checking and Symbolic Computation

EU project to stimulate cooperations
More than 50 partners and associates

Industry: Altran, BTC, ClearSy, Imandra, L4B, Maplesoft, Microsoft, MJC2, NAG, SRI, Systerel, Wolfram

Also present at ACA'18:
Anna Bigatti, Francisco Botana, James Davenport, Vijay Ganesh, Martin Kreuzer, Antonio Montes, Lorenzo Robbiano, Werner Seiler

Computer Science: SMT solving

Satisfiability Modulo Theories (SMT)

Is an existentially quantified first-order formula φ satisfiable?

$$\exists x.\varphi(x) \equiv true$$

Computer Science: SMT solving

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Applications:

- Software verification, test-case generation
- Termination proving
- Controller synthesis
- Scheduling and planning
- Product design automation
- And growing ...

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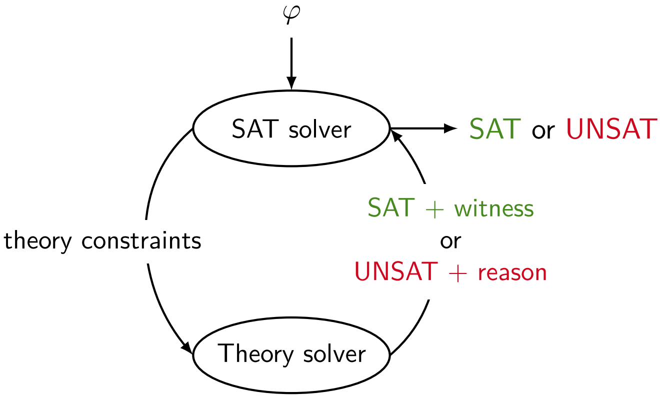
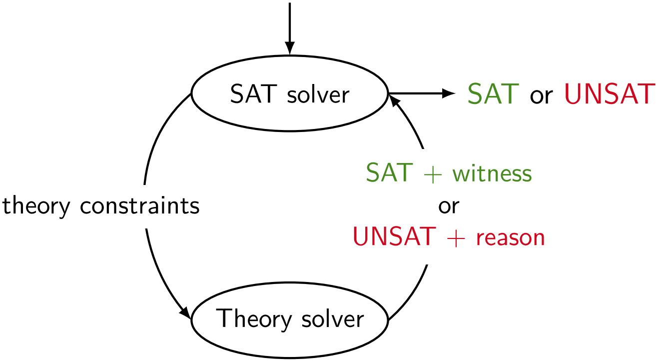
SMT solving



SMT solving

$$x > 0 \wedge (x^2 > 0 \vee x < 0) \wedge (x^3 < 0 \vee x = 3)$$



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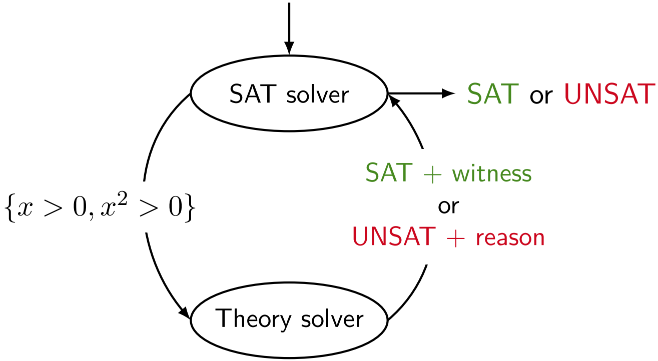

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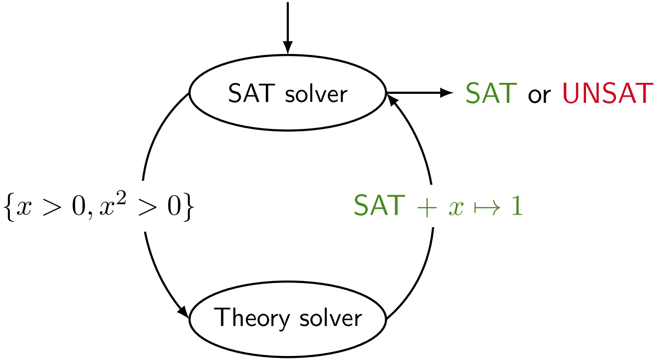

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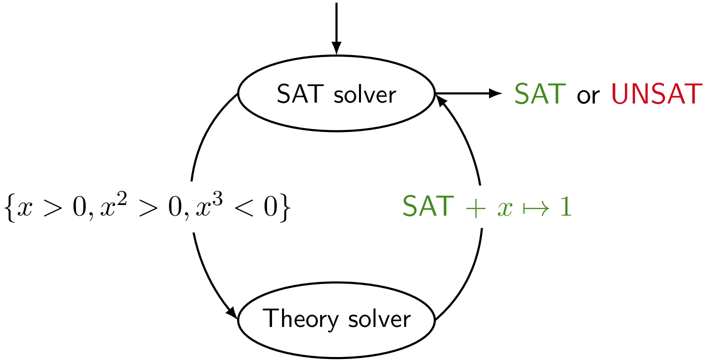

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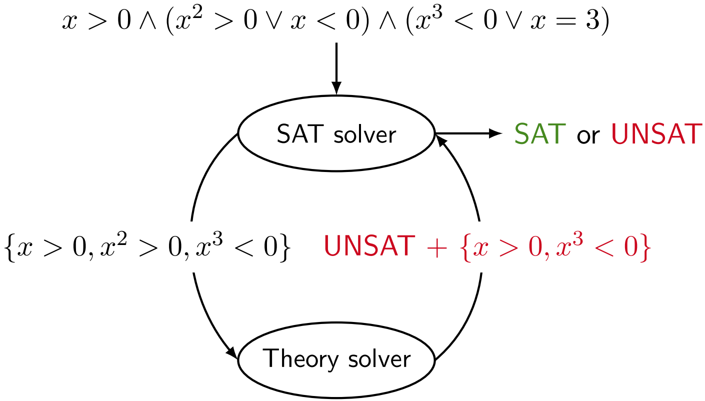
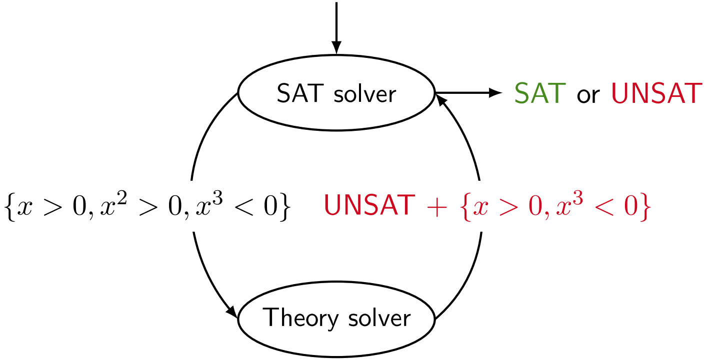

$x > 0 \land (x^2 > 0 \lor x < 0) \land (x^3 < 0 \lor x = 3)$
SAT solver
SAT or UNSAT
$\{x > 0, x^2 > 0, x^3 < 0\}$
UNSAT + $\{x > 0, x^3 < 0\}$
Theory solver

SMT solving

$$x > 0 \wedge (x^2 > 0 \vee x < 0) \wedge (x^3 < 0 \vee x = 3) \wedge (\neg x > 0 \vee \neg x^3 < 0)$$



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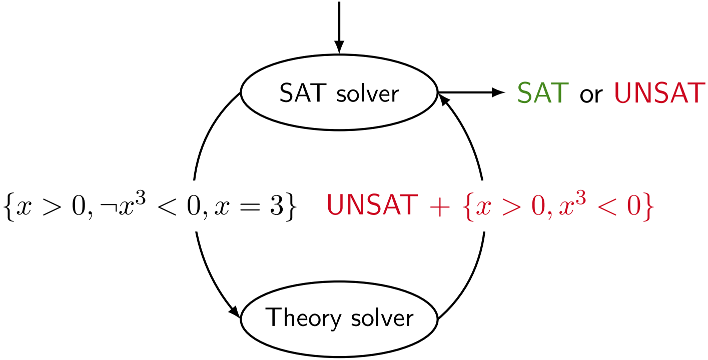

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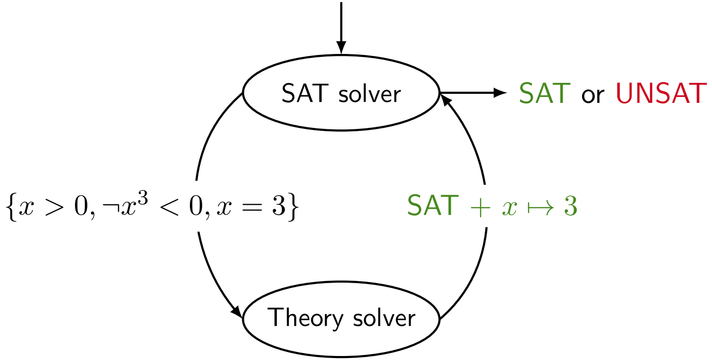

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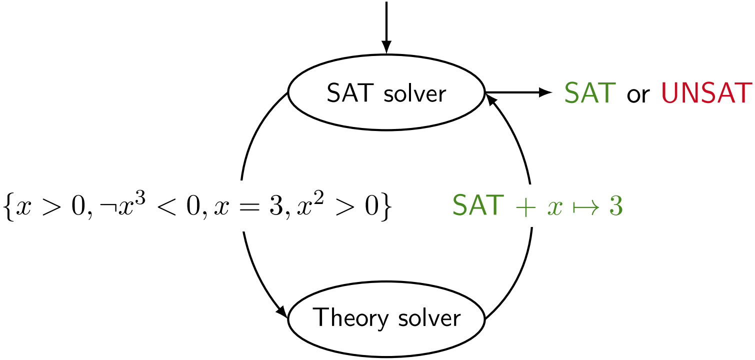

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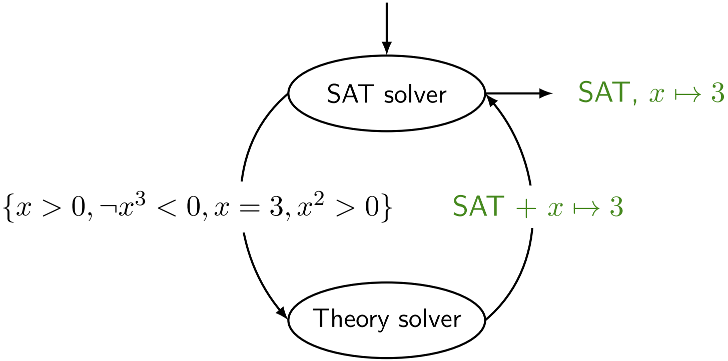

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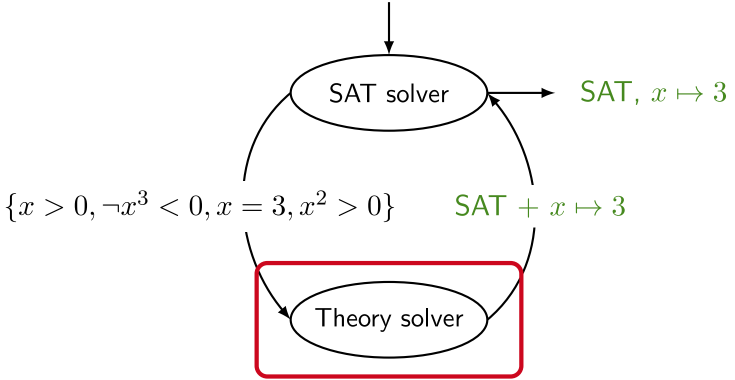

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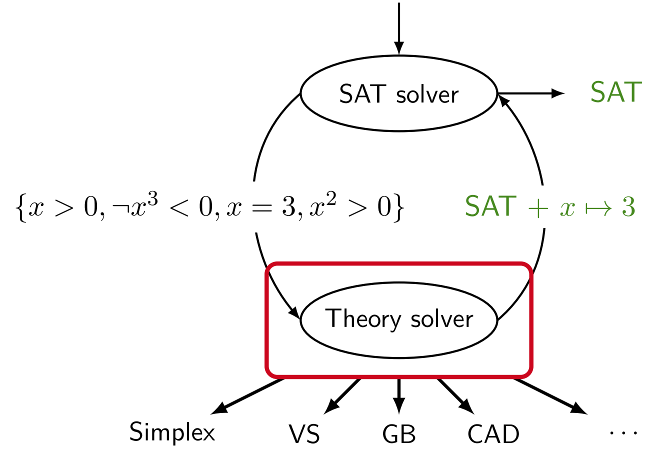

Our solver: SMT-RAT [CKJ+15]

Toolbox for SMT solving

- Modular framework to combine solving techniques
- Various solving modules: SAT, Simplex, ICP, GB, VS, CAD, ...
- Strategic combination to build an SMT solver
- Low-threshold platform for experiments

Aimed at: `QF_NRA`, `QF_NIA`, `QF_PB`
Also supported: `QF_LRA`, `QF_LIA`, `QF_RDL`, `QF_IDL`, `QF_BV`

See `https://github.com/smtrat/smtrat`

Theory solvers

Nonlinear problems are difficult, but you know how to tackle them.

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Properties we like (SMT compliancy)

- Automatable (push-button solution)
- Preferably complete, at least fail verbosely
- Satisfying witness
- Reason for unsatisfiability (infeasible subset)
- Input can be extended (incrementality)
- Input can be reduced (backtracking)

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SMT compliancy – what we can do

- Automation
- Early abort
- Adapt method to our application
  Effective heuristics, low-end modifications, preprocessing, ...
- Provide (reasonably) efficient implementations
- Apply our solutions to industrial problems

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  Effective heuristics, low-end modifications, preprocessing, ...
- Provide (reasonably) efficient implementations
- Apply our solutions to industrial problems
- Incorporate incrementality and backtracking
  Gröbner Bases [JLCA13], CAD [CKJ+15, Hae17]
- Reasons for unsatisfiability [JLCA13, Hen17]
- Combine solving techniques [CKJ+15]

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Success stories

- Virtual Substitution as theory solver [CA11, KCA16]
  Incrementality and backtracking, reasons for unsatisfiability, support for integer problems

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- NLSAT: novel CAD-based solving scheme [JDM12]
  Uses CAD to construct single cells

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Custom implementations for all of this?

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Seriously?

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Custom implementations for all of this?

Seriously?

Yes.

Using other software

What works well (for us):

- `GMP`, `Eigen`
- Originally used `GiNaC` and `CLN`, not anymore
- Some functions from `CoCoALib`
  `gcd()`, `factor()`, `squareFreePart()`
- Finding symmetries using `bliss`

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Common Problems:

- Usable `C` / `C++` interface
- Performance
- Conversion overhead
- SMT compliancy

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Gröbner Bases from `CoCoALib` [AB]

`CoCoALib` is dedicated to computing Gröbner Bases.

Open problems:

- Approximate real radical (work in progress, quality vs. speed)
- Backtracking (snapshots?)
- Satisfying witness
- Reason for unsatisfiability ($\rightarrow$ `GenRepr`, expensive?)

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We do not need a Gröbner Basis.
We need an answer to a theory query.

And we guess a Gröbner Basis could provide this answer...

Maple as a theory solver

Maple is better at everything...

```
solve()
```

```
RootFinding:-WitnessPoints()
```

```
RegularChains:-CylindricalAlgebraicDecompose()
```

```
RegularChains:-LazyRealTriangularize()
```

Maple as a theory solver

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`solve()`

The standard solution, unfortunately not suitable here:

- No satisfying witness, „just" a simplified set of constraints
- No information if no solution exists (`NULL`)
- May be incomplete (`_SolutionsMayBeLost`)
- Result may leave theory ($x < 3/y$, $x = \sqrt{2}$, ...)

`RootFinding:-WitnessPoints()`

`RegularChains:-CylindricalAlgebraicDecompose()`

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Maple as a theory solver

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`solve()`

`RootFinding:-WitnessPoints()`

Numeric approach to find solutions of equalities or inequalities

- No way to combine equalities and inequalities
- No support for weak inequalities
- Rounding errors? Reasons for unsatisfiability?

`RegularChains:-CylindricalAlgebraicDecompose()`

`RegularChains:-LazyRealTriangularize()`

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Maple as a theory solver

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`solve()`

`RootFinding:-WitnessPoints()`

`RegularChains:-CylindricalAlgebraicDecompose()`

Essentially the same approach as our own implementation
No early abort, incrementality or backtracking
$\rightarrow$ comparably slow

`RegularChains:-LazyRealTriangularize()`

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Maple as a theory solver

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`solve()`

`RootFinding:-WitnessPoints()`

`RegularChains:-CylindricalAlgebraicDecompose()`

`RegularChains:-LazyRealTriangularize()`

Some early abort compared to `CylindricalAlgebraicDecompose`
Still no incrementality or backtracking
Subject of future investigation

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This is alright for an interactive system.
It must be taken care of in a fully automated one.

What we need help with

- Gröbner Bases for problems on $\mathbb{R}$?
- Satisfying witnesses from Gröbner bases?
- Stability of numerical approaches?
- Guarantees on rounding errors?
- Factorization?
- Multivariate GCD?

Conclusions

You create amazing mathematics.

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- We (want to) use your methods...
- ... but have somewhat peculiar requirements ...
- ... and end up re-implementing a lot.

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Also:

- You can use our software (Maple does)
- We can provide benchmarks

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References

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