Incremental CAD

Making CAD work for SMT solving

based on [KA18]

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ICMS'18 – University of Notre Dame

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Context: SC^2

EU H2020-FETOPEN-2015-CSA 712689

Satisfiability Checking and Symbolic Computation

EU project to stimulate cooperations
More than 50 partners and associates

Industry: Altran, BTC, ClearSy, Imandra, L4B, Maplesoft, Microsoft, MJC2, NAG, SRI, Systerel, Wolfram

Also at ICMS: Erika Ábrahám, James Davenport, Matthew England, Stephen Forrest, Xiao-Shan Gao, Jürgen Gerhard, Jan Horacek, Martin Kreuzer, Alexei Lisitsa, Thomas Sturm

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SMT solving

Satisfiability Modulo Theories (SMT)

Is an existentially quantified first-order formula φ satisfiable?

$$\exists x.\varphi(x) \equiv true$$

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Applications:

- Software verification, test-case generation
- Termination proving
- Controller synthesis
- Scheduling and planning
- Product design automation
- And growing ...

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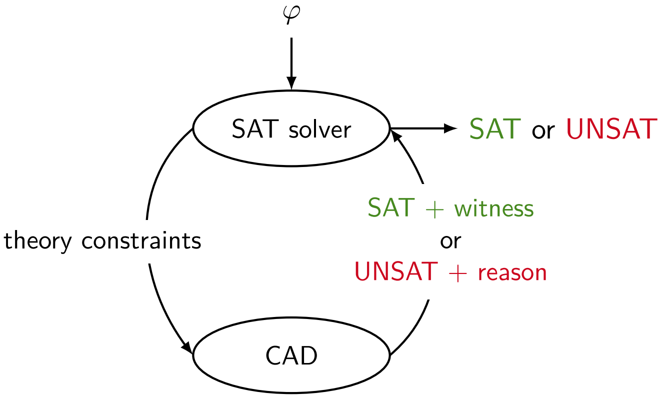
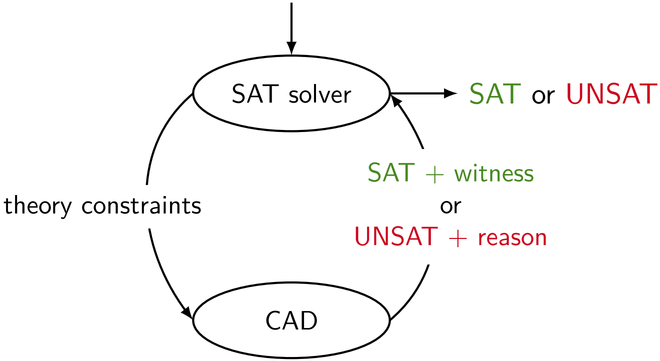
SMT solving



SMT solving

$$x > 0 \land (x^2 > 0 \lor x < 0) \land (x^3 < 0 \lor x = 3)$$



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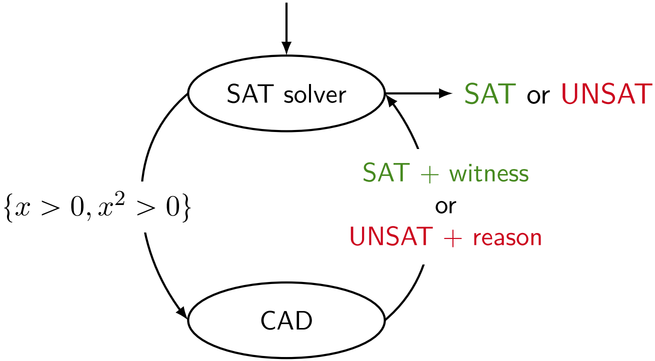

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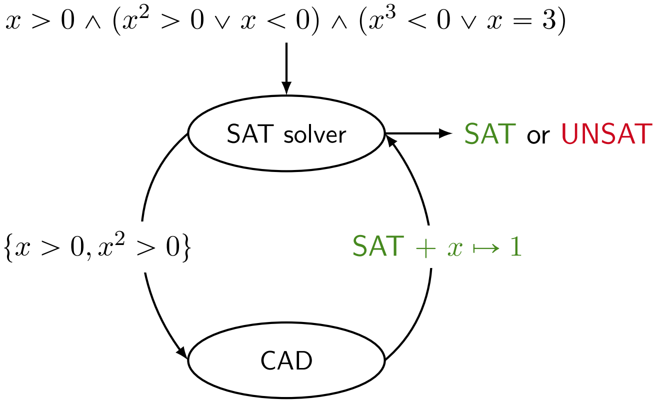

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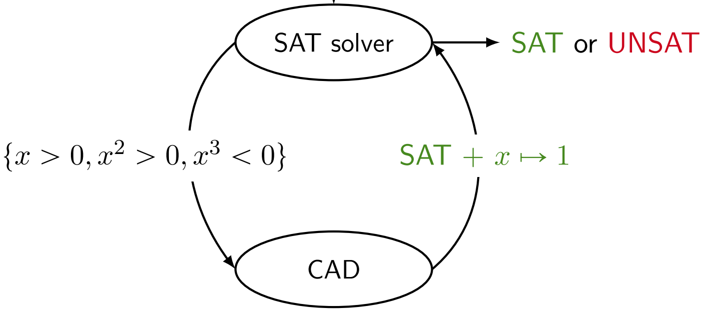

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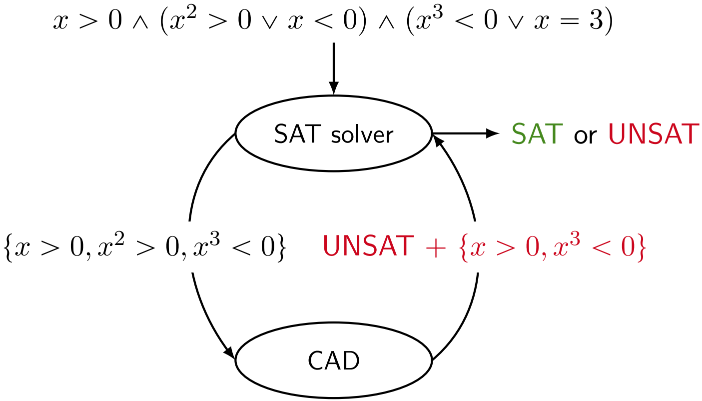

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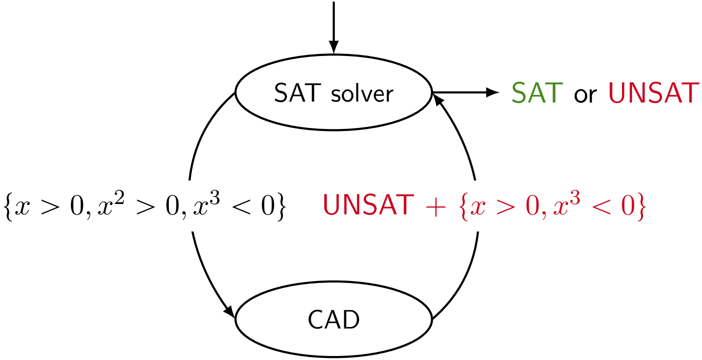

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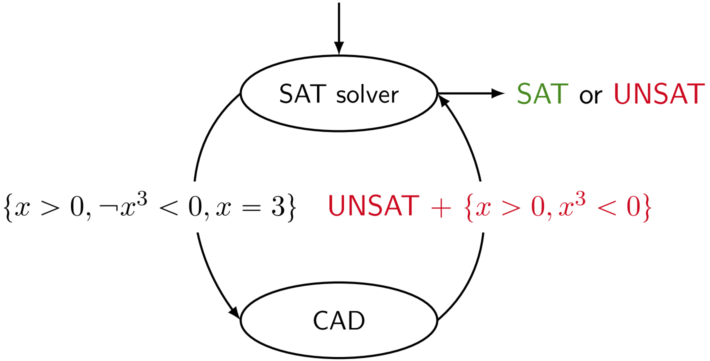

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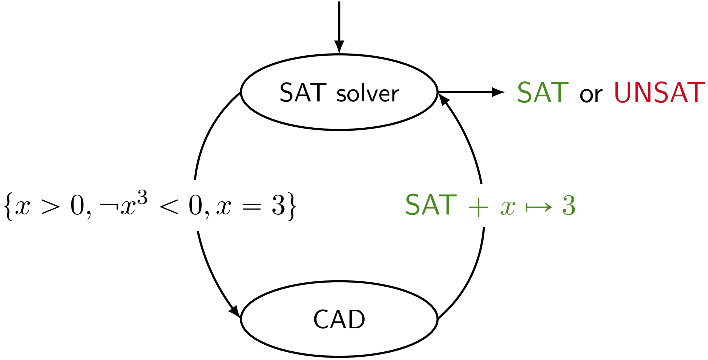

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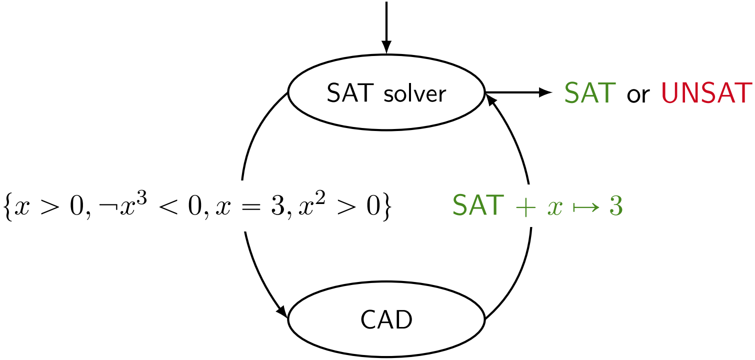

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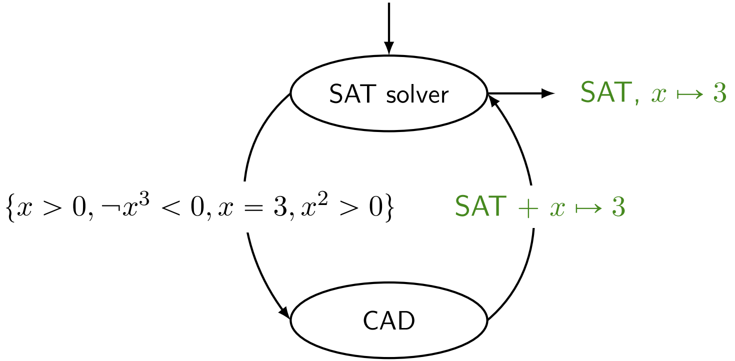

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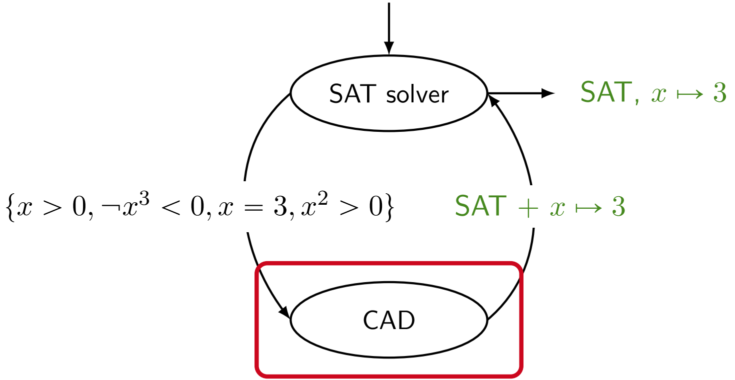

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Cylindrical Algebraic Decomposition

Constraints

$\downarrow$

$P_n \subset \mathbb{Z}[x_1..x_n]$

$\downarrow$

$P_{n-1} \subset \mathbb{Z}[x_1..x_{n-1}]$

$\downarrow$

$\vdots$

$\downarrow$

$P_1 \subset \mathbb{Z}[x_1] \longrightarrow Z_1 \subset \mathbb{R}$

$\uparrow$

$Z_2 \subset Z_1 \times \mathbb{R}$

$\uparrow$

$\vdots$

$\uparrow$

$Z_n \subset Z_{n-1} \times \mathbb{R}$

$\uparrow$

Knowledge

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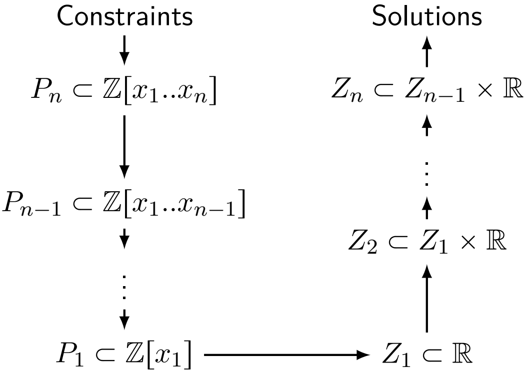
Cylindrical Algebraic Decomposition



Note: We only deal with the purely existential case!

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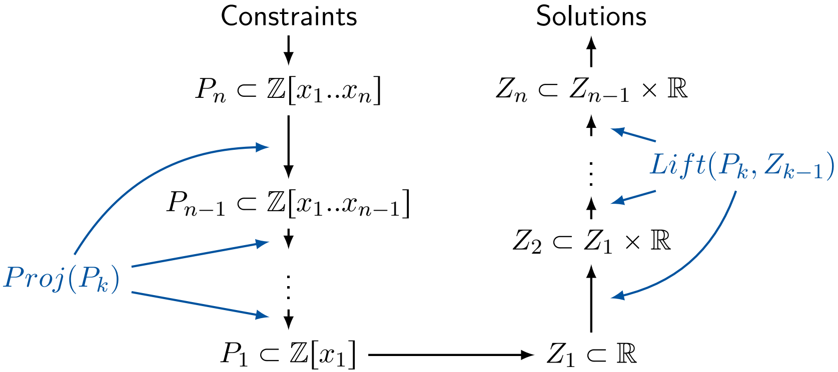

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SMT compliancy

- Stop early when a satisfying witness is found
- Add constraints and check again
- Remove constraints
- Provide reason for unsatisfiability

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  How to get there fast?
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  - How to get there fast?
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  - How to remove some of the polynomials and samples?
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- Remove constraints
  - How to remove some of the polynomials and samples?
  - How to throw away as little information as possible?
- Provide reason for unsatisfiability
  - Which constraints reject all the samples?
  - Solved in [JDF15], though implementation differs.

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Taking a step back

What is the purpose of CAD for us?

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CAD – Traditional approach [Col75]

- Extract P_n from constraints
- For $k = n \dots 2$: $P_{n-1} = Proj(P_n)$
- Sample Z_1 from P_1
- For $k = 2 \dots n$: $Z_k = Lift(P_n, Z_{n-1})$
- Extract solutions from Z_n

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- For $k = 2 \dots n$: $Z_k = Lift(P_n, Z_{n-1})$
- Extract solutions from Z_n

- We compute all polynomials
- We compute all sample points
- We have no idea how to add or remove constraints

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Partial CAD [CH91]

Observations:

- Every $s \in Z_{k-1}$ can be lifted separately.
- Every $s \in Z_{k-1}$ induces a separate set $Z_k^s \subseteq Z_k$.

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Essential idea

Consider the lifting to be a tree of sample points.
Explore the tree recursively.
Evaluate partial sample points during exploration.
Eagerly propagate evaluation results and skip redundant sample points.

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Essential idea

Consider the lifting to be a tree of sample points.
Explore the tree recursively.
Evaluate partial sample points during exploration.
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Additionally:

- Lift $s \in Z_{k-1}$ with every $p \in P_k$ separately.
- Keep a queue of remaining lifting steps (s, p) for continuation.

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Partial projection

Rough template for recent projection operators [McC98, Bro01]

$$Proj(P) = \{\mathrm{disc}(p), \mathrm{coeffs}^*(p) \mid p \in P\} \cup \\ \{\mathrm{res}(p,q) \mid p,q \in P\}$$

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Observations:

- Every step is local to only one or two polynomials.
- We can interrupt the computation frequently.

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Observations:

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- We can interrupt the computation frequently.

Key ideas:

- Split $Proj(P)$ into a sequence of projection steps.
- Keep a queue of projection steps for continuation.

Lazy projection

Observations:

- We lift every (s, p) individually.
- We can also lift $(s, \cdot)$ and guess.
- We can stop as soon as a satisfying sample point is found.

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Why should we even start with the projection?

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Why should we even start with the projection?

Key ideas:

- Start with lifting.
- Perform lifting with respect to an incomplete projection.
- Only when lifting is complete*, spend time on the projection.

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Lazy partial CAD

1. Perform lifting.
2. Return SAT if satisfying sample point was found.
3. Return UNSAT if the projection is complete.
4. Perform a projection step, go back to 1.

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Lazy partial CAD

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Observations:

- Completely driven by the search for a solution.
- Eventually converges to a complete CAD (if UNSAT).
- New choices to be made:
  - Order of lifting steps? (DFS? BFS? Something else?)
  - Order of projection steps? (Level? Degree?)
- Can be continued easily.

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Back to our topic

What about SMT compliancy now?

Reminder:

- ✓ early abort
- add constraints
- remove constraints
- ✓ reasons for unsatisfiability

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Adding constraints

Observations:

- Partial projection maintains a queue of remaining projection steps.
- Partial lifting maintains a queue of remaining lifting steps.
- We can extend these queues for a new polynomial.
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Adding constraints

Observations:

- Partial projection maintains a queue of remaining projection steps.
- Partial lifting maintains a queue of remaining lifting steps.
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If partial projection and partial lifting is in place...

- ... adding new polynomials is easy.
- ... extending the partial CAD is natural.
- ... all previous computations can be reused.

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Removing constraints

Our lazy partial CAD is monotically growing.
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  How to do that?

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Backtracking in Projection

Add $z^2 + y^2 + x^2 < 4$

$$z^2 + y^2 + x^2 < 4$$

z

y

x

Backtracking in Projection

Add $z^2 + y^2 + x^2 - 4$

$$z^2 + y^2 + x^2 < 4$$

z	$z^2 + y^2 + x^2 - 4$
y	
x	

Backtracking in Projection

Project $z^2 + y^2 + x^2 - 4$

$$z^2 + y^2 + x^2 < 4$$

z: $z^2 + y^2 + x^2 - 4$

y: $y^2 + x^2 - 4$

x:

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Backtracking in Projection

Project $y^2 + x^2 - 4$

$$z^2 + y^2 + x^2 < 4$$

z: $z^2 + y^2 + x^2 - 4$

↓

y: $y^2 + x^2 - 4$

↓

x: $x^2 - 4$

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Backtracking in Projection

Add $z^2 < 1$

$z^2 < 1$ $\qquad$ $z^2 + y^2 + x^2 < 4$

z: $z^2 + y^2 + x^2 - 4$

↓

y: $y^2 + x^2 - 4$

↓

x: $x^2 - 4$

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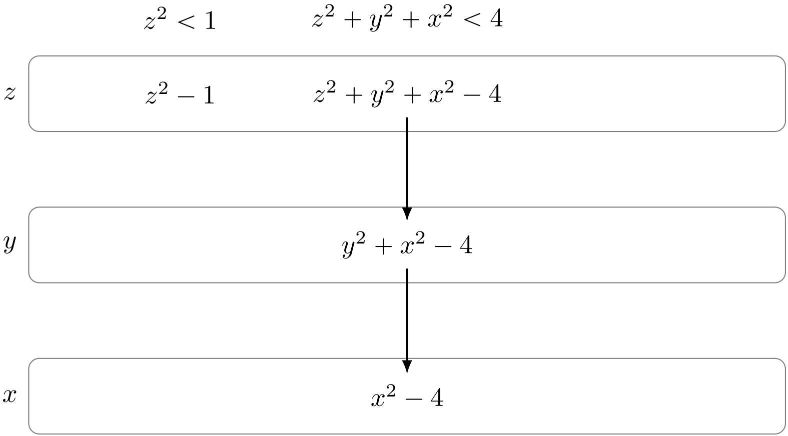
Backtracking in Projection

Add $z^2 - 1$



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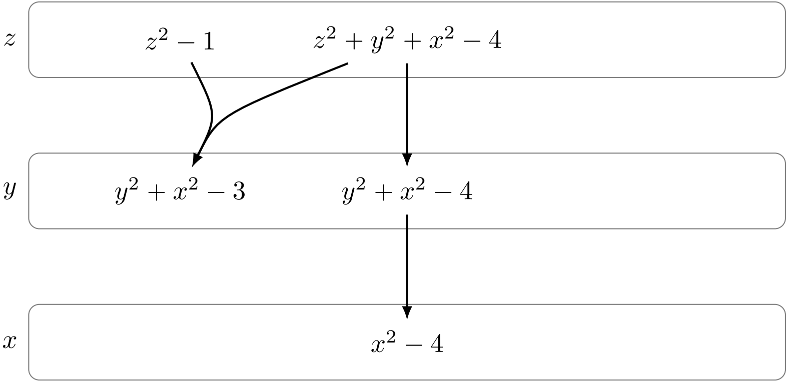
Backtracking in Projection

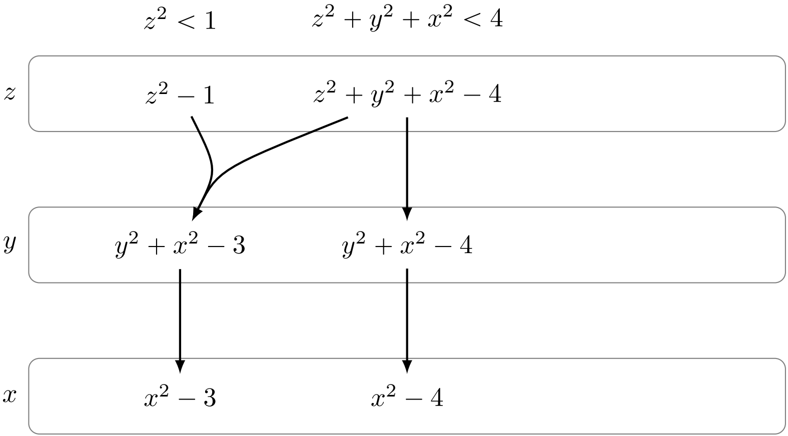
Project $z^2 - 1$ and $z^2 + y^2 + x^2 - 4$

$z^2 < 1$ $\quad$ $z^2 + y^2 + x^2 < 4$



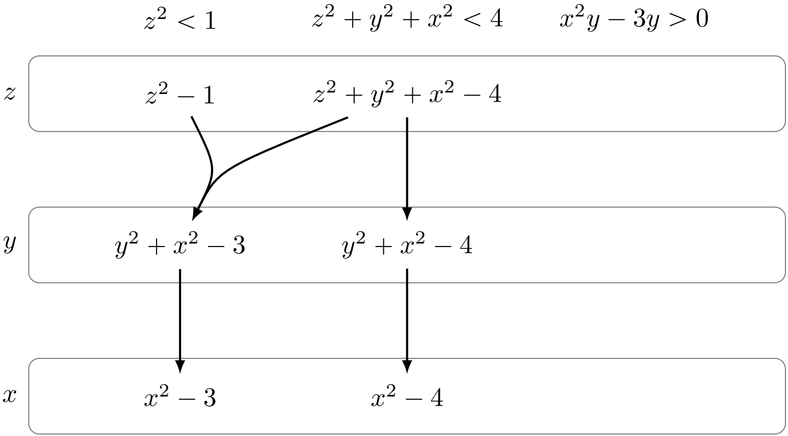
Backtracking in Projection

Project $y^2 + x^2 - 3$



Backtracking in Projection

Add $x^2y - 3y > 0$



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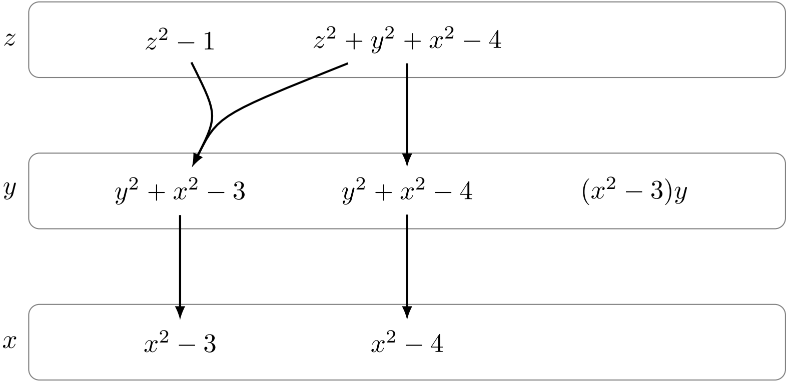
Backtracking in Projection

Add $(x^2 - 3)y$

$z^2 < 1$ $\quad$ $z^2 + y^2 + x^2 < 4$ $\quad$ $x^2y - 3y > 0$



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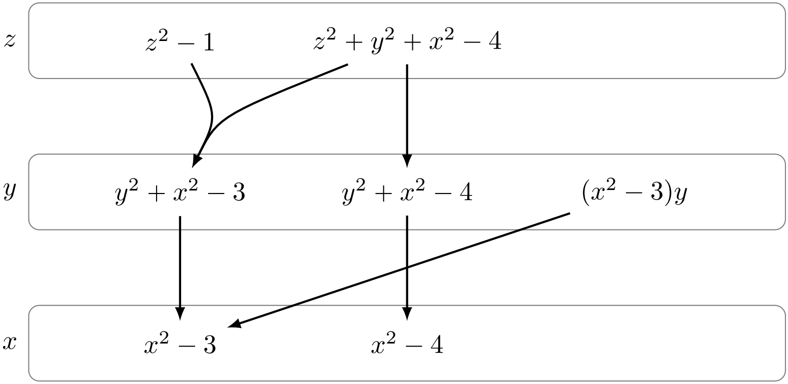

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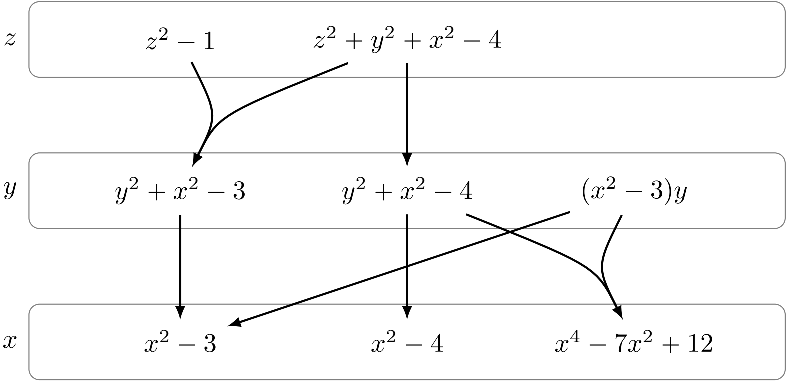
Backtracking in Projection

Project $y^2 + x^2 - 4$ and $(x^2 - 3)y$

$z^2 < 1$ $\quad$ $z^2 + y^2 + x^2 < 4$ $\quad$ $x^2y - 3y > 0$



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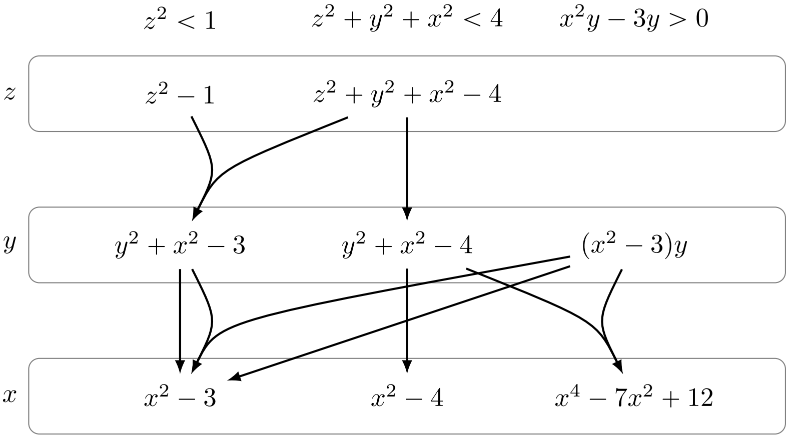
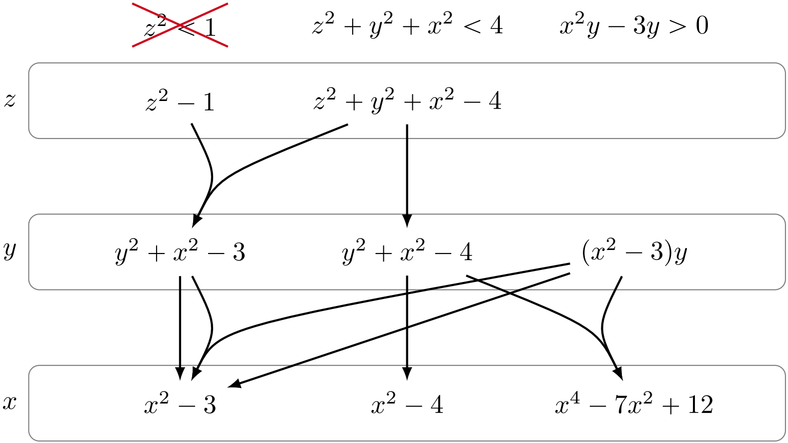
Backtracking in Projection

Project $y^2 + x^2 - 3$ and $(x^2 - 3)y$



Backtracking in Projection

Remove $z^2 < 1$



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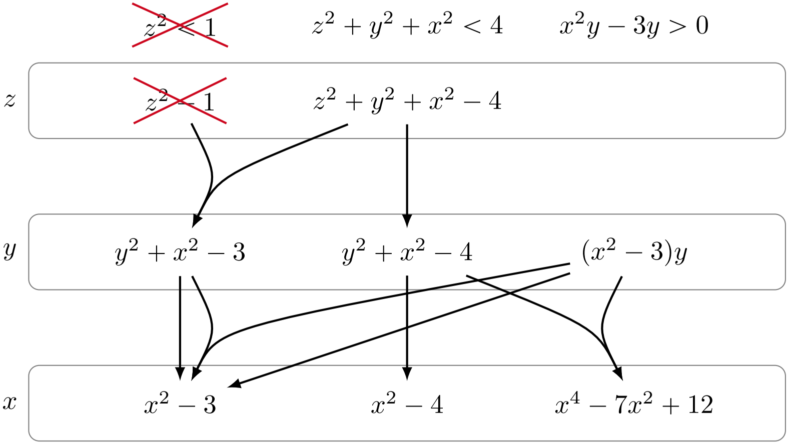
Backtracking in Projection

Remove $z^2 - 1$



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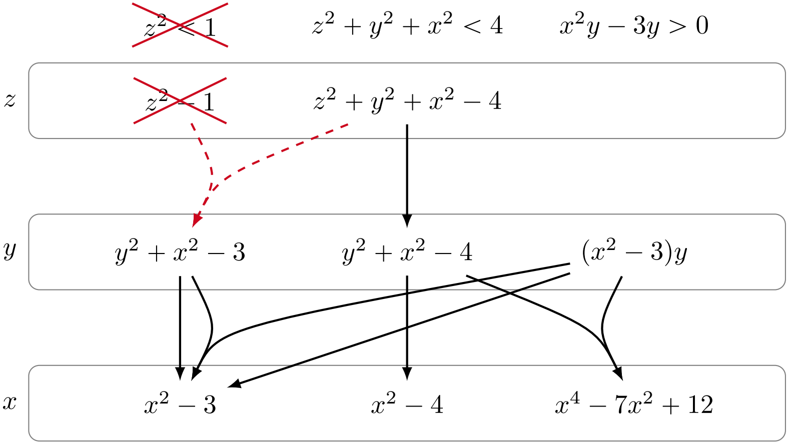
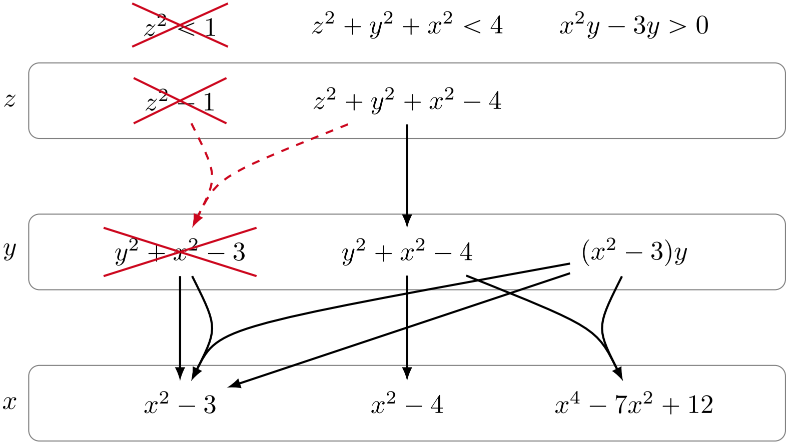


Backtracking in Projection

Remove $y^2 + x^2 - 3$



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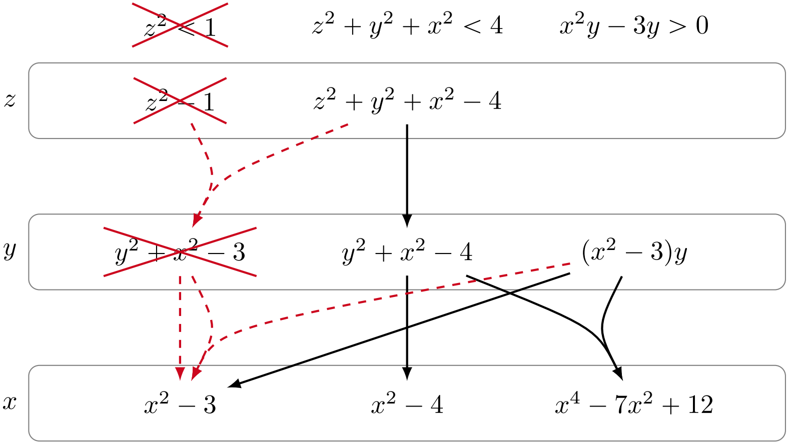

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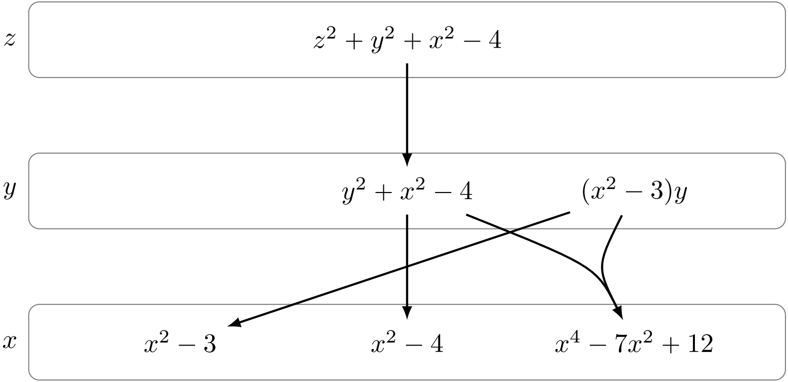
Backtracking in Projection

$$z^2 + y^2 + x^2 < 4 \qquad x^2y - 3y > 0$$



Pruning the lifting tree

Prune lifting after polynomials are removed.

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Prune lifting after polynomials are removed.

Observations:

- A sample is either
  - a root of one or more polynomial(s) or
  - a value in-between two roots.
- We store the reasons for a root as a set of polynomials.

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Observations:

- A sample is either
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- We store the reasons for a root as a set of polynomials.

To prune:

- Remove a root if the reasons are gone.
- Remove one of the neighboring samples with every root.

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Evaluation

Is it worth it?

Experiments

- Benchmarks from SMT-LIB `QF_NRA` (11354 from 10 sources)
- Only `SAT` + `CAD` with different options:

Experiments

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- Only `SAT` $+$ `CAD` with different options:
  - CAD_{Naive}: Fresh CAD on every theory call
  - CAD_{Eager}: Eagerly compute full projection
  - CAD_{Simple}: Eagerly add one constraint at a time
  - CAD_{Full}: Incremental projection

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Experiments

- Benchmarks from SMT-LIB `QF_NRA` (11354 from 10 sources)
- Only `SAT` + `CAD` with different options:
  - CAD_{Naive}: Fresh CAD on every theory call
  - CAD_{Eager}: Eagerly compute full projection
  - CAD_{Simple}: Eagerly add one constraint at a time
  - CAD_{Full}: Incremental projection

Solver	**solved**		**runtime**
CAD_{Naive}	5571	49.1 %	0.69
CAD_{Eager}	7559	66.6 %	0.60
CAD_{Simple}	7924	69.8 %	1.11
CAD_{Full}	8158	71.9 %	1.22

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Conclusion

- Consider CAD as search method for a satisfying solution.
- Perform projection and lifting incrementally.

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- Very beneficial for practical solving.
- More details in [KA18]

Further topics

- Factorization of polynomials?
  Integrates easily, only slight improvement
- Equational constraints?
  Somewhat tricky, only slight improvements [Hae17, HKÁ18]
- Impact of different heuristics?
  Surprisingly small, as long as we exploit incrementality
- Delineating polynomials?
  Integrates easily, somewhat obsolete (?)
- Generic quantifier elimination?
  Disable early abort and obtain a full CAD.
- Implementation?
  Bookkeeping is somewhat involved, but not to bad. See SMT-RAT!

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