Techniques for NRA in SMT

How to solve Nonlinear Real Arithmetic

... and a lot of references

Contains mostly other people's work!

Contains joint work with: Erika Ábrahám, Florian Corzilius, James Davenport, Matthew England, Rebecca Haehn, Jasper Nalbach

Satisfiability modulo theories

Let's skip that...

SMT for Nonlinear Real Arithmetic

Here: Theory of the Reals

SMT for Nonlinear Real Arithmetic

Here: Theory of the Reals

Nonlinear Real Arithmetic:

- real variables $v := x_i \in \mathbb{R}$
- constants $c := q \in \mathbb{Z}$
- terms $t := v \mid c \mid t + t \mid t \cdot t$
- atoms $a := t \sim 0$, $\sim \in \{<, >, \leq, \geq, =, \neq\}$

SMT for Nonlinear Real Arithmetic

Here: Theory of the Reals

Nonlinear Real Arithmetic:

- real variables $v := x_i \in \mathbb{R}$
- constants $c := q \in \mathbb{Z}$
- terms $t := v \mid c \mid t + t \mid t \cdot t$
- atoms $a := t \sim 0$, $\sim \in \{<, >, \leq, \geq, =, \neq\}$

Intuition: polynomials over real variables compared to zero.

SMT for Nonlinear Real Arithmetic

Here: Theory of the Reals

Nonlinear Real Arithmetic:

- real variables $v := x_i \in \mathbb{R}$
- constants $c := q \in \mathbb{Z}$
- terms $t := v \mid c \mid t + t \mid t \cdot t$
- atoms $a := t \sim 0$, $\sim \in \{<, >, \leq, \geq, =, \neq\}$

Intuition: polynomials over real variables compared to zero.

Does cover: $t > t$, rational constants, division (encoding with auxiliary variables)
Does not cover: transcendental constants, non-polynomial functions

SMT for Nonlinear Real Arithmetic

Here: Theory of the Reals

Nonlinear Real Arithmetic:

- real variables $v := x_i \in \mathbb{R}$
- constants $c := q \in \mathbb{Z}$
- terms $t := v \mid c \mid t + t \mid t \cdot t$
- atoms $a := t \sim 0$, $\sim \in \{<, >, \leq, \geq, =, \neq\}$

Intuition: polynomials over real variables compared to zero.

Does cover: $t > t$, rational constants, division (encoding with auxiliary variables)
Does not cover: transcendental constants, non-polynomial functions

Linear arithmetic: essentially a solved problem.
Use Simplex (or sometimes Fourier-Motzkin)

Theory of the Reals in a nutshell

- complete (we have decision procedures that are sound and complete)
- admits quantifier elimination (quantifiers are conceptually easy)

Theory of the Reals in a nutshell

- complete (we have decision procedures that are sound and complete)
- admits quantifier elimination (quantifiers are conceptually easy)

Some methods:

- [Tarski 1951] Tarski: first complete method, non-elementary complexity
- [Buchberger 1965] Gröbner bases: limited applicability, standard tool in CA
- [Collins 1974] CAD: complete, doubly exponential complexity
- [Weispfenning 1988] VS: up to bounded degree, singly exponential complexity
- [Gao et al. 2013] ICP: heuristic interval reasoning, incomplete
- [Fontaine et al. 2017] Subtropical satisfiability: incomplete reduction to LRA
- [Irfan 2018] Linearization: incomplete, axiom instantiation
- [Ábrahám et al. 2021] CDCAC: conflict-driven CAD
- and some more...

Overview

1. SMT for NRA
2. Linearization
3. Interval Constraint Propagation
4. Subtropical Satisfiability
5. Gröbner Bases
6. Virtual Substitution
7. Cylindrical Algebraic Decomposition
8. Conflict-Driven Cylindrical Algebraic Coverings
9. Related topics

Linearization by example

[Irfan 2018] [Cimatti et al. 2018]

- Linearize atoms
- Solve
- Identify conflicts
- Instantiate axioms
- Add as lemmas
- Repeat

Linearization by example

[Irfan 2018] [Cimatti et al. 2018]

- Linearize atoms
- Solve
- Identify conflicts
- Instantiate axioms
- Add as lemmas
- Repeat

$$x \cdot y \leq 0 \land x < 0 \land x + y = 0$$

$$\text{linearize: } z \leq 0 \land x < 0 \land x + y = 0 \qquad z := x \cdot y$$

$$\text{atoms: } z \leq 0 \land x < 0 \land x + y = 0$$

$$\text{solve: } x \mapsto -1, y \mapsto 1, z \mapsto 0$$

$$\text{conflict: } 0 \neq -1 \cdot 1$$

$$\text{axiom: } z = 0 \Rightarrow (x = 0 \lor y = 0)$$

add axiom as lemma, proceed to next theory call

$$\text{atoms: } z \leq 0 \land x < 0 \land x + y = 0 \land z \neq 0$$

$$\text{solve: } x \mapsto -1, y \mapsto 1, z \mapsto -1$$

SAT!

Linearization

[Irfan 2018] [Cimatti et al. 2018]

Intuition: iteratively teach the linear solver about the nonlinear parts, add lemmas that cut away unsatisfiable regions.

Linearization

[Irfan 2018] [Cimatti et al. 2018]

Intuition: iteratively teach the linear solver about the nonlinear parts, add lemmas that cut away unsatisfiable regions.

More axioms: zeroes, monotonicity, commutativity, symmetry w.r.t. signs, tangent planes, . . .

Linearization

[Irfan 2018] [Cimatti et al. 2018]

Intuition: iteratively teach the linear solver about the nonlinear parts, add lemmas that cut away unsatisfiable regions.

More axioms: zeroes, monotonicity, commutativity, symmetry w.r.t. signs, tangent planes, ...

Problems: difficult to identify models (linear solver only finds corners), linear solver only finds rational assignments ($x^2 = 2$)

Linearization

[Irfan 2018] [Cimatti et al. 2018]

Intuition: iteratively teach the linear solver about the nonlinear parts, add lemmas that cut away unsatisfiable regions.

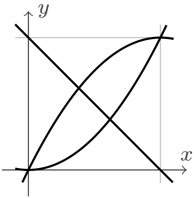
More axioms: zeroes, monotonicity, commutativity, symmetry w.r.t. signs, tangent planes, ...

Problems: difficult to identify models (linear solver only finds corners), linear solver only finds rational assignments ($x^2 = 2$)

Extensions:

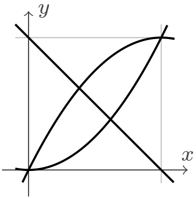
- Repair model (if easily possible)
- Transcendental functions (`sin`, `cos`, ...)
- extended operators in general

Interval Constraint Propagation by example



$$y > x^2 \;\wedge\; y < -x^2 + 2x \;\wedge\; y \leq 1 - x \qquad x \times y$$

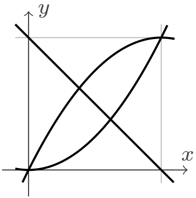
Interval Constraint Propagation by example



$y > x^2 \quad \wedge \quad y < -x^2 + 2x \quad \wedge \quad y \leq 1 - x$ $\qquad x \times y$

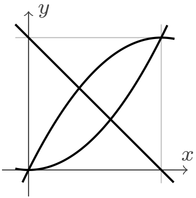
$y > x^2 \Rightarrow y \in (0, \infty)$ $\qquad (-\infty, \infty) \times (0, \infty)$

Interval Constraint Propagation by example



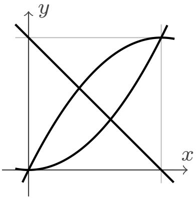
$y > x^2 \quad \wedge \quad y < -x^2 + 2x \quad \wedge \quad y \leq 1 - x$	$x \times y$
$y > x^2 \Rightarrow y \in (0, \infty)$	$(-\infty, \infty) \times (0, \infty)$
$x > 0.5x^2 + y \Rightarrow x \in (0, \infty)$	$(0, \infty) \times (0, \infty)$

Interval Constraint Propagation by example



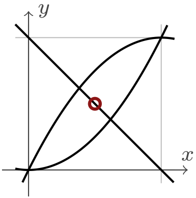
$y > x^2 \;\wedge\; y < -x^2 + 2x \;\wedge\; y \leq 1 - x$	$x \times y$
$y > x^2 \Rightarrow y \in (0, \infty)$	$(-\infty, \infty) \times (0, \infty)$
$x > 0.5x^2 + y \Rightarrow x \in (0, \infty)$	$(0, \infty) \times (0, \infty)$
$x \leq -y + 1 \Rightarrow x \in (0, 1)$	$(0, 1) \times (0, \infty)$

Interval Constraint Propagation by example



$y > x^2 \quad \wedge \quad y < -x^2 + 2x \quad \wedge \quad y \leq 1 - x$	$x \times y$
$y > x^2 \Rightarrow y \in (0, \infty)$	$(-\infty, \infty) \times (0, \infty)$
$x > 0.5x^2 + y \Rightarrow x \in (0, \infty)$	$(0, \infty) \times (0, \infty)$
$x \leq -y + 1 \Rightarrow x \in (0, 1)$	$(0, 1) \times (0, \infty)$
$y \leq -x + 1 \Rightarrow y \in (0, 1)$	$(0, 1) \times (0, 1)$

Interval Constraint Propagation by example



$y > x^2 \quad \wedge \quad y < -x^2 + 2x \quad \wedge \quad y \leq 1 - x$	$x \times y$
$y > x^2 \Rightarrow y \in (0, \infty)$	$(-\infty, \infty) \times (0, \infty)$
$x > 0.5x^2 + y \Rightarrow x \in (0, \infty)$	$(0, \infty) \times (0, \infty)$
$x \leq -y + 1 \Rightarrow x \in (0, 1)$	$(0, 1) \times (0, \infty)$
$y \leq -x + 1 \Rightarrow y \in (0, 1)$	$(0, 1) \times (0, 1)$
guess midpoint $(0.5, 0.5) \in (0, 1) \times (0, 1)$	

Interval Constraint Propagation in a nutshell

[Benhamou et al. 2006] [Gao et al. 2013] [Scheibler et al. 2013] [Schupp 2013] [Tung et al. 2017]

Core idea:

- Maintain interval assignment (that represents the current box)
- Perform over-approximating contractions until
  - the current box is empty (UNSAT),
  - we can guess a model (SAT), or
  - we reach a threshold.
- When reaching a threshold
  - we terminate with unknown or
  - split: $x \in [0, 5] \rightsquigarrow (x < 3 \vee x \geq 3)$

Interval Constraint Propagation in a nutshell

[Benhamou et al. 2006] [Gao et al. 2013] [Scheibler et al. 2013] [Schupp 2013] [Tung et al. 2017]

Core idea:

- Maintain interval assignment (that represents the current box)
- Perform over-approximating contractions until
  - the current box is empty (UNSAT),
  - we can guess a model (SAT), or
  - we reach a threshold.
- When reaching a threshold
  - we terminate with unknown or
  - split: $x \in [0, 5] \rightsquigarrow (x < 3 \vee x \geq 3)$

- Incomplete solving procedure
- Used as preprocessor for other techniques [Loup et al. 2013]
- Delicate tuning of heuristics (splitting, thresholds, model guessing)

Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

- Assume $p(1, \ldots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \ldots, 1)$ — x

Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

- Assume $p(1, \ldots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \ldots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

- Assume $p(1, \dots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \dots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Core problem: How to find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$?

Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

- Assume $p(1, \dots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \dots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Core problem: How to find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$?

For $n = 1$: $\lim_{x \to \infty} p(x) = \infty$ if $\text{lcoeff}(p) > 0$. Increase x as necessary.

Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

- Assume $p(1, \dots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \dots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Core problem: How to find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$?

For $n = 1$: $\lim_{x \to \infty} p(x) = \infty$ if $\operatorname{lcoeff}(p) > 0$. Increase x as necessary.
For $n \geq 2$: search direction in exponent space such that the largest exponent in this direction is positive. Increase x in this direction as necessary.

Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

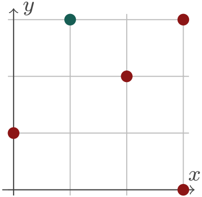
Core idea: reduce $p = 0$ to a linear problem in the exponents of p

- Assume $p(1, \dots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \dots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Core problem: How to find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$?

For $n = 1$: $\lim_{x \to \infty} p(x) = \infty$ if $\mathrm{lcoeff}(p) > 0$. Increase x as necessary.
For $n \geq 2$: search direction in exponent space such that the largest exponent in this direction is positive. Increase x in this direction as necessary.

$$p = -y + 2xy^3 - 3x^2y^2 - x^3 - 4x^3y^3$$



Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

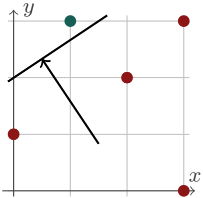
- Assume $p(1, \ldots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \ldots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Core problem: How to find $x \in \mathbb{R}^n_+$ such that $p(x) > 0$?

For $n = 1$: $\lim_{x\to\infty} p(x) = \infty$ if $\operatorname{lcoeff}(p) > 0$. Increase x as necessary.
For $n \geq 2$: search direction in exponent space such that the largest exponent in this direction is positive. Increase x in this direction as necessary.

$$p = -y + 2xy^3 - 3x^2y^2 - x^3 - 4x^3y^3$$

Find hyperplane that separates a positive node



Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

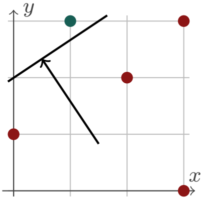
- Assume $p(1, \ldots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \ldots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Core problem: How to find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$?

For $n = 1$: $\lim_{x \to \infty} p(x) = \infty$ if $\operatorname{lcoeff}(p) > 0$. Increase x as necessary.
For $n \geq 2$: search direction in exponent space such that the largest exponent in this direction is positive. Increase x in this direction as necessary.

$$p = -y + 2xy^3 - 3x^2y^2 - x^3 - 4x^3y^3$$

Find hyperplane that separates a positive node
Encoding in `QF_LRA`
Growing degree only impacts coefficient size



Gröbner basis

[Buchberger 1965] [Junges 2012]

- Canonical generators for a polynomial ideal
- For us: Normal form for sets of polynomials
- Maintains set of common complex roots
- The workhorse of computer algebra for polynomial equalities
- Mature implementations (every CAS)
- Doubly exponential in worst case, but usually much faster.

Gröbner basis

[Buchberger 1965] [Junges 2012]

- Canonical generators for a polynomial ideal
- For us: Normal form for sets of polynomials
- Maintains set of common complex roots
- The workhorse of computer algebra for polynomial equalities
- Mature implementations (every CAS)
- Doubly exponential in worst case, but usually much faster.

Relevant for SMT: $\exists x \in \mathbb{C}^n . p(x) = 0$

Gröbner basis

[Buchberger 1965] [Junges 2012]

- Canonical generators for a polynomial ideal
- For us: Normal form for sets of polynomials
- Maintains set of common complex roots
- The workhorse of computer algebra for polynomial equalities
- Mature implementations (every CAS)
- Doubly exponential in worst case, but usually much faster.

Relevant for SMT: $\exists x \in \mathbb{C}^n . p(x) = 0$

But: What about inequalities? How to go from $\mathbb{C}$ to $\mathbb{R}$?
see [Junges 2012] for some approaches.

Virtual Substitution

[Weispfenning 1988] [Weispfenning 1997] [Košta et al. 2015] [Košta 2016] [Nalbach 2017]

Core idea:

- Use solution formula to solve polynomial equation for x
- Substitute value for x into remaining equations
- Repeat for remaining variables

Virtual Substitution

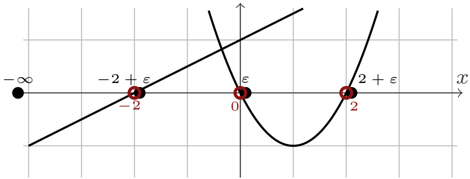
[Weispfenning 1988] [Weispfenning 1997] [Košta et al. 2015] [Košta 2016] [Nalbach 2017]

Core idea:

- Use solution formula to solve polynomial equation for x
- Substitute value for x into remaining equations
- Repeat for remaining variables

What about inequalities?

- Construct test candidates for all sign-invariant regions in x
- Always try the roots and the smallest values of the intermediate intervals



- Introduces special terms $t+\varepsilon$ and $-\infty$

Virtual Substitution

Algorithmic core: a collection of substitution rules
Example: Substitute $e + \varepsilon$ for x into $a \cdot x^2 + b \cdot x + c > 0$:

$$
\begin{array}{llllllll}
& (& (ax^2 + bx + c > 0)[e//x] & & & & &) \\
\vee & (& (ax^2 + bx + c = 0)[e//x] & \wedge & (2ax + b > 0)[e//x] & & &) \\
\vee & (& (ax^2 + bx + c = 0)[e//x] & \wedge & (2ax + b = 0)[e//x] & \wedge & (2a > 0)[e//x] &)
\end{array}
$$

Virtual Substitution

Algorithmic core: a collection of substitution rules
Example: Substitute $e + \varepsilon$ for x into $a \cdot x^2 + b \cdot x + c > 0$:

$$\begin{array}{lllllll} & (& (ax^2+bx+c>0)[e//x] & & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b>0)[e//x] & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b=0)[e//x] & \wedge \quad (2a>0)[e//x] &) \end{array}$$

Not always applicable:

- Solution formulas only exist up to degree four
- The above rule may introduce a degree growth
- Efficient if applicable
- [Košta et al. 2015] uses FO formulas, allows arbitrary but fixed degrees (needs precomputed substitution rules obtained by quantifier elimination)

Cylindrical Algebraic Decomposition

The core idea: sign-invariance (or rather truth-table equivalence)

$$sgn(p(a)) = sgn(p(b))\ \forall p \in \varphi \quad \Rightarrow \quad \varphi(a) = \varphi(b)$$

For our purpose, a and b are equivalent!

Cylindrical Algebraic Decomposition

The core idea: sign-invariance (or rather truth-table equivalence)

$$sgn(p(a)) = sgn(p(b)) \; \forall p \in \varphi \quad \Rightarrow \quad \varphi(a) = \varphi(b)$$

For our purpose, a and b are equivalent!

Construct a sign-invariant decomposition of $\mathbb{R}^n$:

$$\text{cell C} \subset \mathbb{R}^n : \forall a, b \in C : \varphi(a) = \varphi(b)$$

Abstraction: $\mathbb{R}^n$ to finite set of cells, consider a single $a \in C$ per cell.

Cylindrical Algebraic Decomposition

The core idea: sign-invariance (or rather truth-table equivalence)

$$sgn(p(a)) = sgn(p(b)) \; \forall p \in \varphi \quad \Rightarrow \quad \varphi(a) = \varphi(b)$$

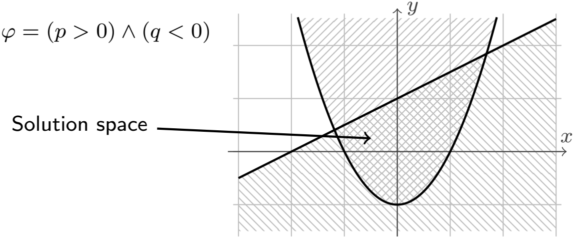
For our purpose, a and b are equivalent!

Construct a sign-invariant decomposition of $\mathbb{R}^n$:

$$\text{cell C} \subset \mathbb{R}^n : \forall a, b \in C : \varphi(a) = \varphi(b)$$

Abstraction: $\mathbb{R}^n$ to finite set of cells, consider a single $a \in C$ per cell.

$$\varphi = (p > 0) \land (q < 0)$$



Cylindrical Algebraic Decomposition

The core idea: sign-invariance (or rather truth-table equivalence)

$$sgn(p(a)) = sgn(p(b)) \; \forall p \in \varphi \quad \Rightarrow \quad \varphi(a) = \varphi(b)$$

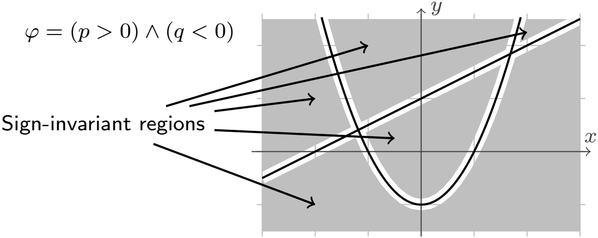
For our purpose, a and b are equivalent!

Construct a sign-invariant decomposition of $\mathbb{R}^n$:

$$\text{cell C} \subset \mathbb{R}^n : \forall a, b \in C : \varphi(a) = \varphi(b)$$

Abstraction: $\mathbb{R}^n$ to finite set of cells, consider a single $a \in C$ per cell.

$\varphi = (p > 0) \wedge (q < 0)$



Cylindrical Algebraic Decomposition

The core idea: sign-invariance (or rather truth-table equivalence)

$$sgn(p(a)) = sgn(p(b)) \ \forall p \in \varphi \quad \Rightarrow \quad \varphi(a) = \varphi(b)$$

For our purpose, a and b are equivalent!

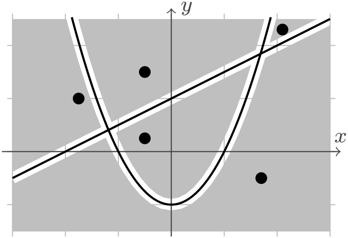
Construct a sign-invariant decomposition of $\mathbb{R}^n$:

$$\text{cell C} \subset \mathbb{R}^n : \forall a, b \in C : \varphi(a) = \varphi(b)$$

Abstraction: $\mathbb{R}^n$ to finite set of cells, consider a single $a \in C$ per cell.

$\varphi = (p > 0) \wedge (q < 0)$

Sample points



Cylindrical Algebraic Decomposition

The core idea: sign-invariance (or rather truth-table equivalence)

$$sgn(p(a)) = sgn(p(b))\ \forall p \in \varphi \quad \Rightarrow \quad \varphi(a) = \varphi(b)$$

For our purpose, a and b are equivalent!

Construct a sign-invariant decomposition of $\mathbb{R}^n$:

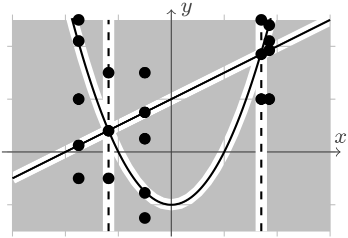
$$\text{cell C} \subset \mathbb{R}^n : \forall a, b \in C : \varphi(a) = \varphi(b)$$

Abstraction: $\mathbb{R}^n$ to finite set of cells, consider a single $a \in C$ per cell.

$\varphi = (p > 0) \wedge (q < 0)$

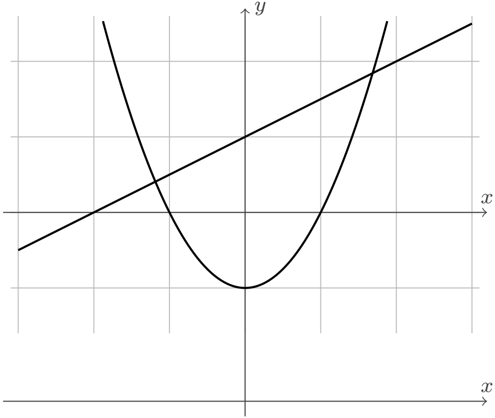
Actual sample points

Arranged in cylinders



Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

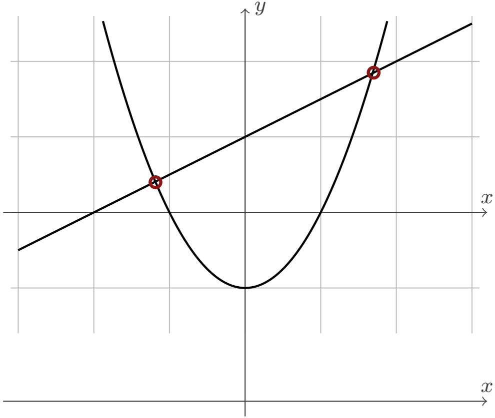


Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

Intuition

Critical points



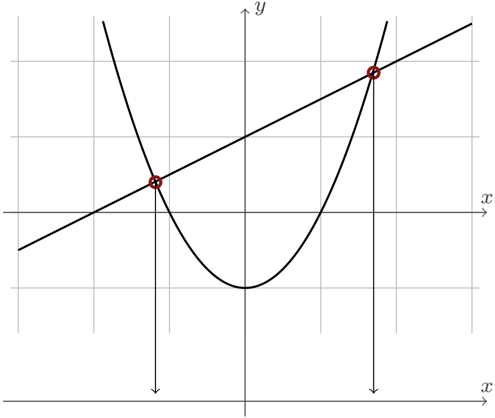
Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

Intuition

Critical points

Project sample



Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

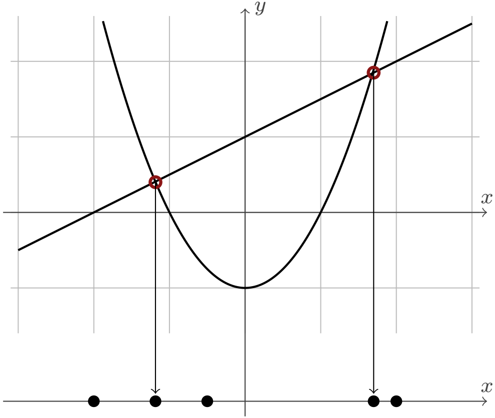
Proceed dimension-wise: project to lower-dimensional problem, lift results.

Intuition

Critical points

Project sample

Solve 1-dim



Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

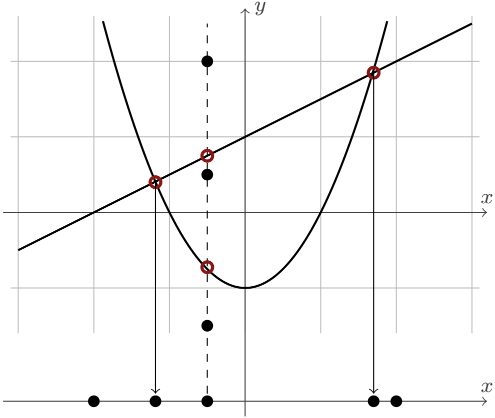
Intuition

Critical points

Project sample

Solve 1-dim

Lift to 2-dim



Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

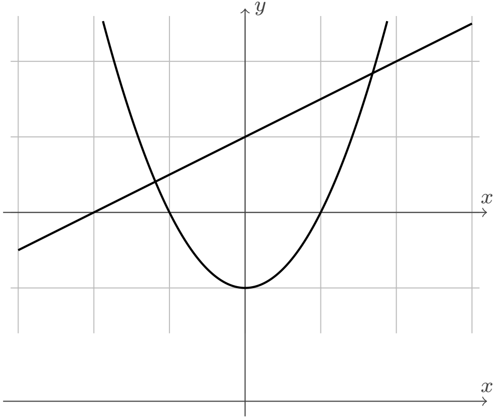
Intuition

Critical points

Project sample

Solve 1-dim

Lift to 2-dim



Implementation

Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

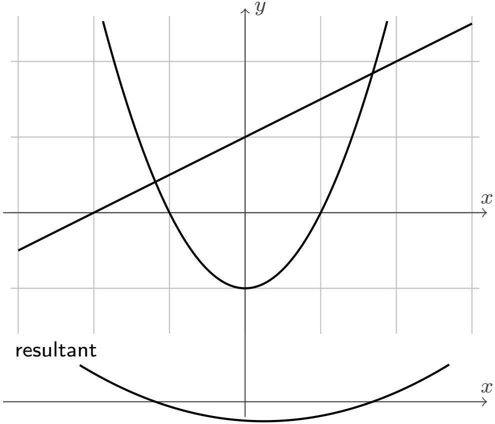
Intuition

Critical points

Project sample

Solve 1-dim

Lift to 2-dim



Implementation

Project polynomials

Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

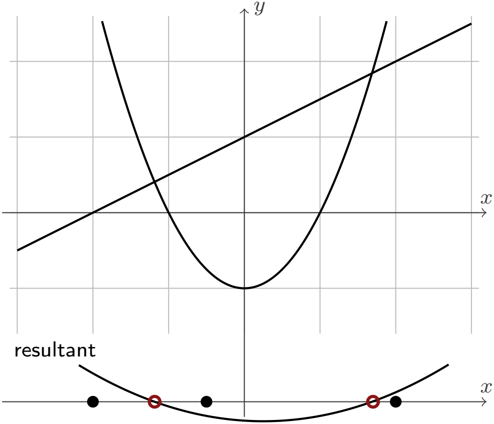
Intuition

Critical points

Project sample

Solve 1-dim

Lift to 2-dim



Implementation

Project polynomials

Solve 1-dim

Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.

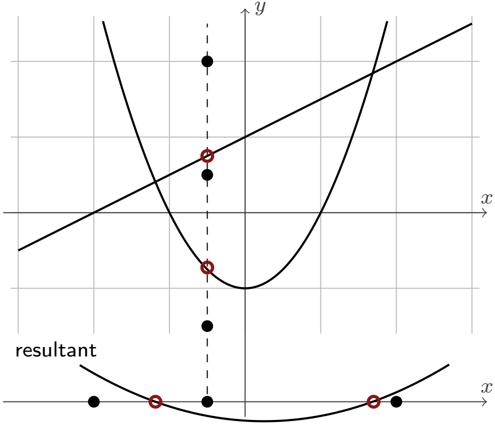
Intuition

Critical points

Project sample

Solve 1-dim

Lift to 2-dim



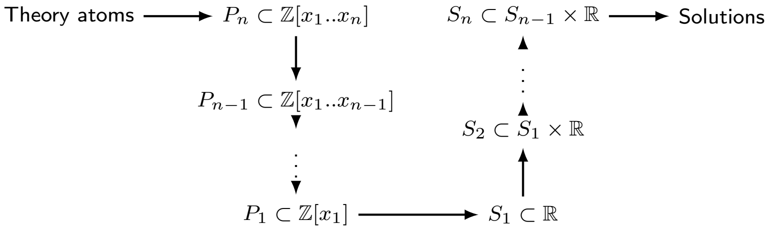
Implementation

Project polynomials

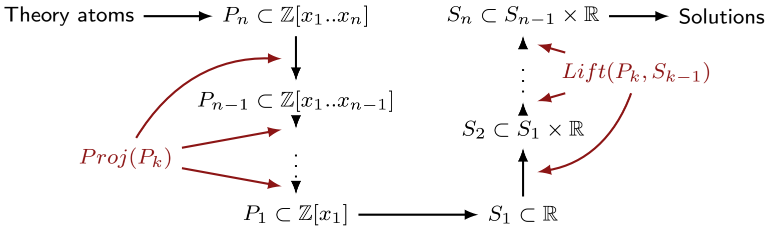
Solve 1-dim

Lift to 2-dim

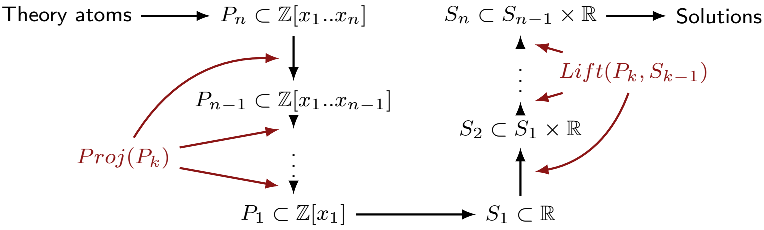
Cylindrical Algebraic Decomposition in $\mathbb{R}^n$



Cylindrical Algebraic Decomposition in $\mathbb{R}^n$

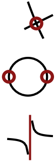


Cylindrical Algebraic Decomposition in $\mathbb{R}^n$

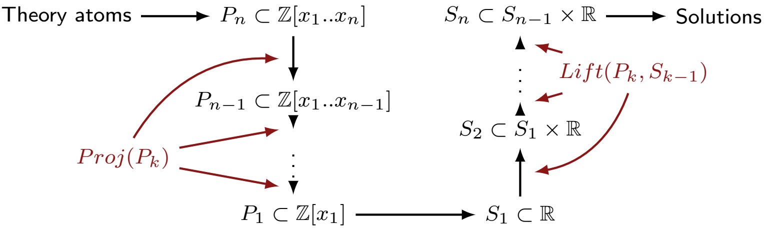


Projection:

- Intersections (resultants)
- Flipping points (discriminants)
- Singularities (coefficients)



Cylindrical Algebraic Decomposition in $\mathbb{R}^n$



Projection:

- Intersections (resultants)
- Flipping points (discriminants)
- Singularities (coefficients)

Lifting:

- Substitution $s \in S_k$, $p \in P_{k+1}$
  $p(s) \to p' \in \mathbb{Z}[x_{k+1}]$ ***
- Isolate real roots of p'

Final notes on CAD

- Asymptotic complexity: $(n \cdot m)^{2^r}$ (r variables, m polynomials of degree n)
- Oftentimes way faster, but worst-case occurs in practice!
- Best complete method that is known and implemented. [Hong 1991]
- Active research:
  - Projection [McCallum 1984] [McCallum 1988] [Hong 1990] [Lazard 1994] [Brown 2001] [McCallum 2001] [McCallum et al. 2016] [McCallum et al. 2019];[Strzeboński 2000] [Seidl et al. 2003] [Jovanović et al. 2012] [Brown 2013] [Strzeboński 2014] [Brown et al. 2015]
  - Lifting [Collins 1974] [Lazard 1994] [McCallum et al. 2016] [McCallum et al. 2019]
  - Equational constraints [Collins 1998] [McCallum 1999] [McCallum 2001] [England et al. 2015] [Haehn et al. 2018] [Nair et al. 2019]
  - Variable ordering [England et al. 2014] [Huang et al. 2014] [Nalbach et al. 2019] [Florescu et al. 2019]
  - Adaptions [Jovanović et al. 2012] [Brown 2013] [Brown 2015] [Ábrahám et al. 2021]
- Implementation needs groundwork: polynomial computation (resultants, multivariate gcd, optionally multivariate factorization), real algebraic numbers (representation, multivariate root isolation)

Conflict-Driven Cylindrical Algebraic Coverings

[Ábrahám et al. 2021]

Core idea: use CAD techniques in a conflict-driven way.

My intuition: MCSAT turned into a theory solver.

Conflict-Driven Cylindrical Algebraic Coverings

[Ábrahám et al. 2021]

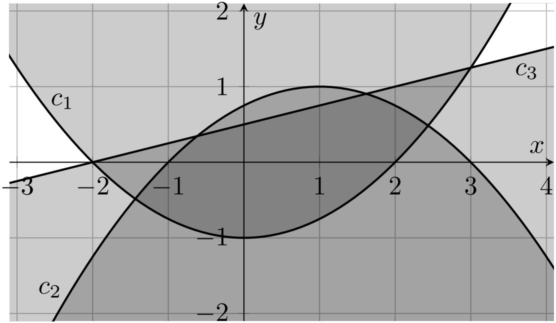
Core idea: use CAD techniques in a conflict-driven way.

My intuition: MCSAT turned into a theory solver.

- Fix a variable ordering
- For the kth variable
  - Use constraints to exclude unsatisfiable intervals
  - Guess a value for the kth variable
  - Recurse to $k+1$st variable and obtain
    - a full variable assignment ($\rightarrow$ return SAT)
    - or a covering for the $k+1$st variable
  - Use CAD machinery to infer an interval for the kth variable
- Until the collected intervals form a covering for the kth variable

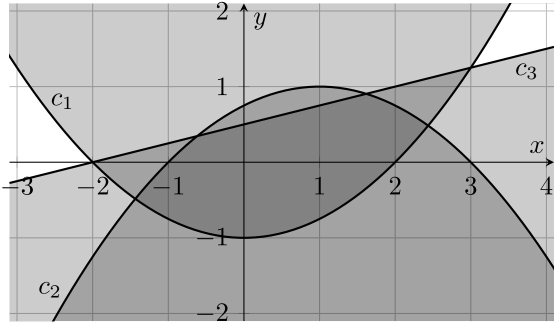
An example

$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$



An example

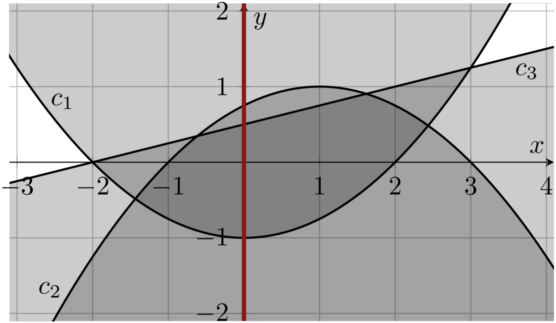
$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$



No constraint for x

An example

$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$

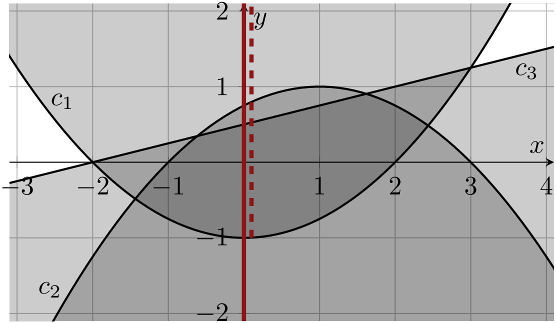


No constraint for x

Guess $x \mapsto 0$

An example

$c_1 : 4 \cdot y < x^2 - 4$ $\qquad$ $c_2 : 4 \cdot y > 4 - (x-1)^2$ $\qquad$ $c_3 : 4 \cdot y > x + 2$



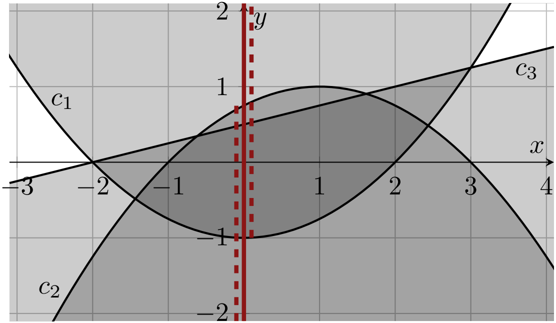
No constraint for x

Guess $x \mapsto 0$

$c_1 \to y \notin (-1, \infty)$

An example

$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$



No constraint for x

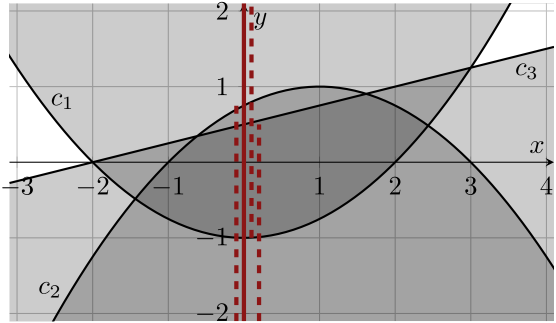
Guess $x \mapsto 0$

$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

An example

$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$



No constraint for x

Guess $x \mapsto 0$

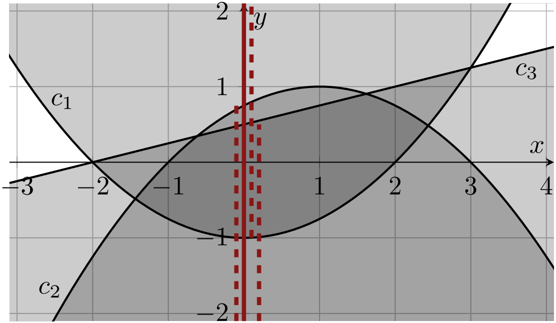
$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

$c_3 \to y \notin (-\infty, 0.5)$

An example

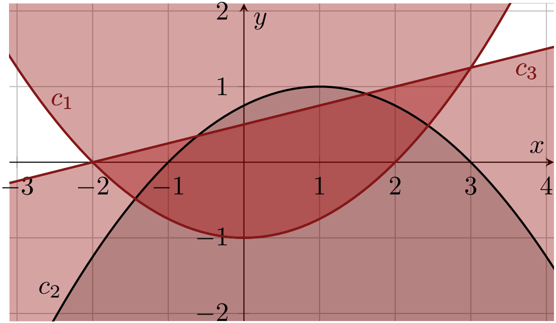
$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$



No constraint for x
Guess $x \mapsto 0$
$c_1 \to y \notin (-1, \infty)$
$c_2 \to y \notin (-\infty, 0.75)$
$c_3 \to y \notin (-\infty, 0.5)$
Construct covering
$(-\infty, 0.5), (-1, \infty)$

An example

$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$



No constraint for x

Guess $x \mapsto 0$

$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

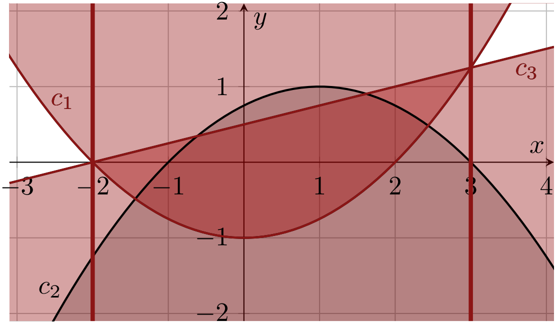
$c_3 \to y \notin (-\infty, 0.5)$

Construct covering

$(-\infty, 0.5), (-1, \infty)$

An example

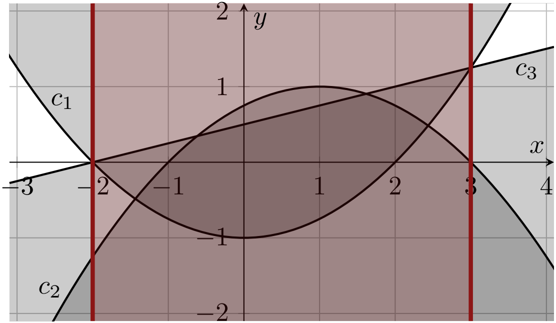
$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$



No constraint for x
Guess $x \mapsto 0$
$c_1 \to y \notin (-1, \infty)$
$c_2 \to y \notin (-\infty, 0.75)$
$c_3 \to y \notin (-\infty, 0.5)$
Construct covering
$(-\infty, 0.5), (-1, \infty)$
Construct interval for x
$x \notin (-2, 3)$

An example

$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$



No constraint for x

Guess $x \mapsto 0$

$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

$c_3 \to y \notin (-\infty, 0.5)$

Construct covering

$(-\infty, 0.5), (-1, \infty)$

Construct interval for x

$x \notin (-2, 3)$

New guess for x

The main algorithm

function get_unsat_cover($(s_1, \ldots, s_{i-1})$)

$\mathbb{I} :=$ get_unsat_intervals(s)
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  $s_i :=$ sample_outside($\mathbb{I}$)
  if $i = n$ **then** **return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  $(f, O) :=$ get_unsat_cover($(s_1, \ldots, s_{i-1}, s_i)$)
  if $f = \mathrm{SAT}$ **then** **return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    $R :=$ construct_characterization($(s_1, \ldots, s_{i-1}, s_i), O$)
    $J :=$ interval_from_characterization($(s_1, \ldots, s_{i-1}), s_i, R$)
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

The main algorithm

```
function get_unsat_cover((s_1, ..., s_{i-1}))

𝕀 := get_unsat_intervals(s)
while ⋃_{I∈𝕀} I ≠ ℝ do
    s_i := sample_outside(𝕀)
    if i = n then return (SAT, (s_1, ..., s_{i-1}, s_i))
    (f, O) := get_unsat_cover((s_1, ..., s_{i-1}, s_i))
    if f = SAT then return (SAT, O)
    else if f = UNSAT then
        R := construct_characterization((s_1, ..., s_{i-1}, s_i), O)
        J := interval_from_characterization((s_1, ..., s_{i-1}), s_i, R)
        𝕀 := 𝕀 ∪ {J}
    end
end
return (UNSAT, 𝕀)
```

Real root isolation over a partial sample point

The main algorithm

```
function get_unsat_cover((s_1, ..., s_{i-1}))

𝕀 := get_unsat_intervals(s)
while ⋃_{I∈𝕀} I ≠ ℝ do
    s_i := sample_outside(𝕀)
    if i = n then return (SAT, (s_1, ..., s_{i-1}, s_i))
    (f, O) := get_unsat_cover((s_1, ..., s_{i-1}, s_i))
    if f = SAT then return (SAT, O)
    else if f = UNSAT then
        R := construct_characterization((s_1, ..., s_{i-1}, s_i), O)
        J := interval_from_characterization((s_1, ..., s_{i-1}), s_i, R)
        𝕀 := 𝕀 ∪ {J}
    end
end
return (UNSAT, 𝕀)
```

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

The main algorithm

function get_unsat_cover($(s_1, \ldots, s_{i-1})$)

$\mathbb{I} :=$ get_unsat_intervals(s)
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  $s_i :=$ sample_outside($\mathbb{I}$)
  if $i = n$ **then** **return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  $(f, O) :=$ get_unsat_cover($(s_1, \ldots, s_{i-1}, s_i)$)
  if $f = \mathrm{SAT}$ **then** **return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    $R :=$ construct_characterization($(s_1, \ldots, s_{i-1}, s_i), O$)
    $J :=$ interval_from_characterization($(s_1, \ldots, s_{i-1}), s_i, R$)
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

The main algorithm

function get_unsat_cover($(s_1, \ldots, s_{i-1})$)

$\mathbb{I} :=$ get_unsat_intervals(s)
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  $s_i :=$ sample_outside($\mathbb{I}$)
  if $i = n$ **then** **return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  $(f, O) :=$ get_unsat_cover($(s_1, \ldots, s_{i-1}, s_i)$)
  if $f = \mathrm{SAT}$ **then** **return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    $R :=$ construct_characterization($(s_1,$
    $J :=$ interval_from_characterization($(s_1,$
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

CAD-style projection: Roots of polynomials restrict where covering is still applicable

The main algorithm

function get_unsat_cover($(s_1, \ldots, s_{i-1})$)

$\mathbb{I}$:= get_unsat_intervals(s)
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  s_i := sample_outside($\mathbb{I}$)
  if $i = n$ **then** **return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  (f, O) := get_unsat_cover($(s_1, \ldots, s_{i-1}, s_i)$)
  if $f = \mathrm{SAT}$ **then** **return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    R := construct_characterization($(s_1$ …
    J := interval_from_characterization($(s_1,$ …
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

CAD-style projection: Roots of polynomials restrict where covering is still applicable

Extract interval from polynomials

The main algorithm

```
function get_unsat_cover((s_1, ..., s_{i-1}))

𝕀 := get_unsat_intervals(s)
while ⋃_{I∈𝕀} I ≠ ℝ do
    s_i := sample_outside(𝕀)
    if i = n then return (SAT, (s_1, ..., s_{i-1}, s_i))
    (f, O) := get_unsat_cover((s_1, ..., s_{i-1}, s_i))
    if f = SAT then return (SAT, O)
    else if f = UNSAT then
        R := construct_characterization((s_1, ...
        J := interval_from_characterization((s_1, ...
        𝕀 := 𝕀 ∪ {J}
    end
end
return (UNSAT, 𝕀)
```

Real root isolation over a partial sample point

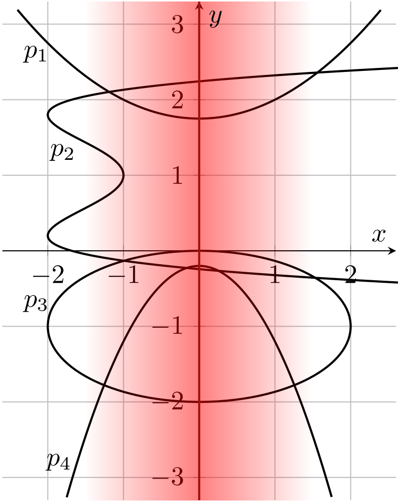
Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

CAD-style projection: Roots of polynomials restrict where covering is still applicable

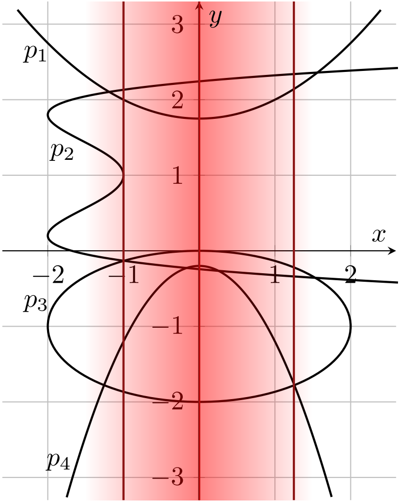
Extract interval from polynomials

construct_characterization



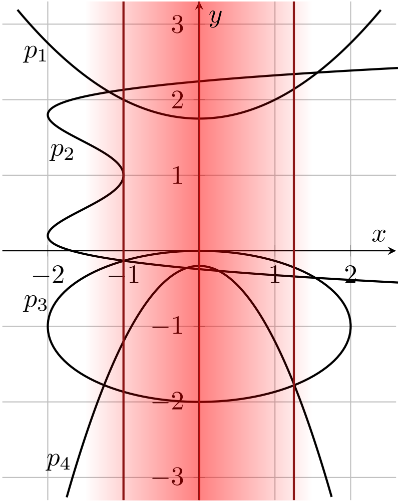
Identify region around sample

construct_characterization



Identify region around sample

construct_characterization

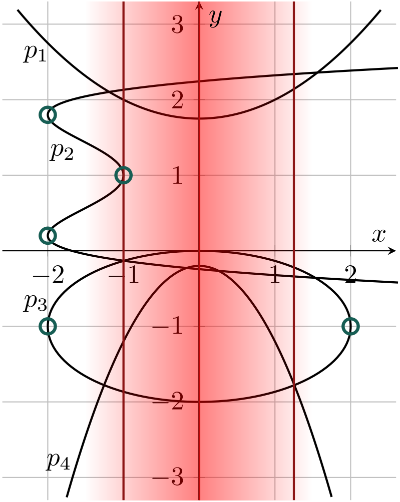


Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

construct_characterization

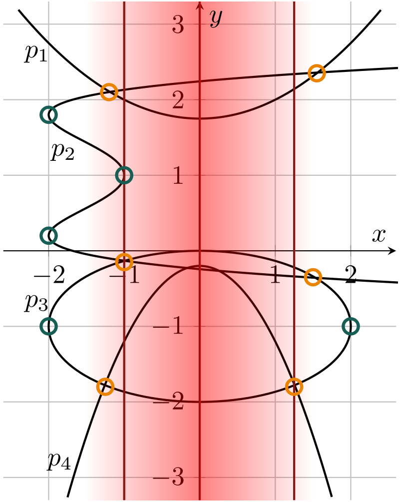


Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

construct_characterization

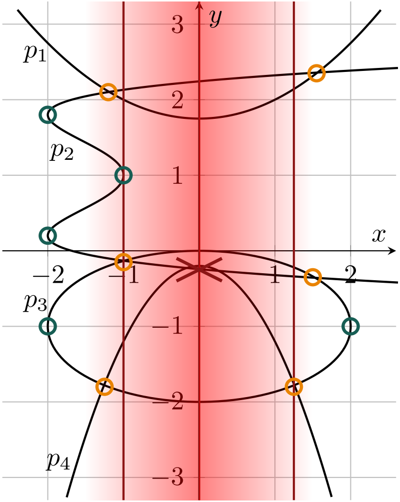


Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

construct_characterization



Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

Improvement over CAD:

- Resultants between neighbouring intervals only!

Other methods for (QF_)NRA

- Numerical methods [Kremer 2013]:
  focus on good approximation, but no formal guarantees
- Tarski's method [Tarski 1951]:
  theoretical breakthrough only, non-elementary complexity
- Grigor'ev and Vorobjov [Grigor'ev et al. 1988], Renegar [Renegar 1988]:
  singly exponentional, but impractical (see [Hong 1991])
- Basu, Pollack and Roy [Basu et al. 1996]:
  "realizable sign conditions", has not been implemented (yet)
- Other CAD-based methods:
  Regular Chains [Chen et al. 2009], NuCAD [Brown 2015]

Beyond QF_NRA

- Quantifiers:
  - Theory of the Reals admits quantifier elimination
  - CAD constructs φ' for $Q_x \varphi(x, y) \Leftrightarrow \varphi'(y)$
- Theory combination with Array, BV, FP, String, ... [Nelson et al. 1979]
- Transcendentals: extend linearization [Cimatti et al. 2018] [Irfan 2018]
- Optimization: CAD can optimize for an objective [Kremer 2020]
- Integers: Branch&Bound complements BitBlasting [Kremer et al. 2016]

Beyond CDCL(T)-style SMT

Other approaches for (QF_)NRA:

- MCSAT / NLSAT:
  - Theory model construction integrated in the core solver
  - SMT-RAT, yices, z3 [Jovanović et al. 2012] [Jovanović et al. 2013] [Moura et al. 2013] [Nalbach et al. 2019] [Kremer 2020]
- CAD is a stand-alone tool:
  - Maple / RegularChains [Chen et al. 2009]
  - Mathematica [Strzeboński 2014]
  - QEPCAD B [Brown 2003]
  - Redlog / Reduce [Dolzmann et al. 1997]

These can be integrated as theory solvers [Fontaine et al. 2018] [Kremer 2018]

cvc5

[Barrett et al. 2011]

- SMT solver developed at Stanford University & University of Iowa
- Supports a wide variety of theories (and their combinations)
  Arithmetic (linear, non-linear, transcendentals), Arrays, Bags & Sets, Bit-vectors, Datatypes, Floating-point, Separation logic, Strings, Uninterpreted functions
- Also Quantifiers, Syntax-Guided Synthesis [Reynolds et al. 2019], UNSAT Cores, verifiable Proofs

cvc5

[Barrett et al. 2011]

- SMT solver developed at Stanford University & University of Iowa
- Supports a wide variety of theories (and their combinations)
  Arithmetic (linear, non-linear, transcendentals), Arrays, Bags & Sets, Bit-vectors, Datatypes, Floating-point, Separation logic, Strings, Uninterpreted functions
- Also Quantifiers, Syntax-Guided Synthesis [Reynolds et al. 2019], UNSAT Cores, verifiable Proofs

obtain cvc5 from

`https://cvc4.github.io/downloads.html` or
`https://github.com/cvc5/cvc5`

cvc5 for QF_NRA

- Linearization (`--nl-ext`)
- CDCAC (`--nl-cad`)
- Also: ICP-style propagations (`--nl-icp`)

cvc5 for QF_NRA

- Linearization (`--nl-ext`)
- CDCAC (`--nl-cad`)
- Also: ICP-style propagations (`--nl-icp`)

Default strategy: incremental linearization with a small subset of the axioms and CDCAC

cvc5 for QF_NRA

- Linearization (--nl-ext)
- CDCAC (--nl-cad)
- Also: ICP-style propagations (--nl-icp)

Default strategy: incremental linearization with a small subset of the axioms and CDCAC

Experiments on QF_NRA (11489 in total)

	solved	sat	unsat
cvc5	10634	5001	5633
yices 2.6.2	10341	4904	5437
z3 4.8.10	10288	5093	5195

cvc5

in progress / future work:

- Better integration of Linearization, CDCAC and ICP
- Preprocessing for nonlinear arithmetic
- Improve proofs
- Improve incrementality (in particular CDCAC)
- Improvements within CDCAC (heuristics, factorization, . . .)

References I

- Erika Ábrahám, James H. Davenport, Matthew England, and Gereon Kremer. "Deciding the Consistency of Non-Linear Real Arithmetic Constraints with a Conflict Driven Search Using Cylindrical Algebraic Coverings". In: Journal of Logical and Algebraic Methods in Programming 119 (2021), p. 100633. DOI: 10.1016/j.jlamp.2020.100633.
- Clark Barrett, Christopher L. Conway, Morgan Deters, Liana Hadarean, Dejan Jovanović, Tim King, Andrew Reynolds, and Cesare Tinelli. "CVC4". In: CAV. Vol. 6806. July 2011, pp. 171–177. DOI: 10.1007/978-3-642-22110-1_14.
- Saugata Basu, Richard Pollack, and Marie-Françoise Roy. "On the Combinatorial and Algebraic Complexity of Quantifier Elimination". In: Journal of the ACM 43 (6 1996), pp. 1002–1045. DOI: 10.1145/235809.235813.
- Frédéric Benhamou and Laurent Granvilliers. "Continuous and Interval Constraints". In: Handbook of Constraint Programming. Vol. 2. 2006, pp. 571–603. DOI: 10.1016/S1574-6526(06)80020-9.
- Christopher W. Brown. "Improved Projection for Cylindrical Algebraic Decomposition". In: Journal of Symbolic Computation 32 (5 2001), pp. 447–465. DOI: 10.1006/jsco.2001.0463.
- Christopher W. Brown. "Qepcad b: A program for computing with semi-algebraic sets using CADs". In: ACM SIGSAM Bulletin 37 (4 2003), pp. 97–108. doi: 10.1145/968708.968710.
- Christopher W. Brown. "Constructing a Single Open Cell in a Cylindrical Algebraic Decomposition". In: ISSAC. 2013, pp. 133–140. doi: 10.1145/2465506.2465952.
- Christopher W. Brown. "Open Non-uniform Cylindrical Algebraic Decompositions". In: ISSAC. 2015, pp. 85–92. doi: 10.1145/2755996.2756654.

References II

- Christopher W. Brown and Marek Košta. "Constructing a single cell in cylindrical algebraic decomposition". In: Journal of Symbolic Computation 70 (2015), pp. 14–48. doi: 10.1016/j.jsc.2014.09.024.
- Bruno Buchberger. "Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal". PhD thesis. University of Innsbruck, 1965.
- Changbo Chen, Marc Moreno Maza, Bican Xia, and Lu Yang. "Computing Cylindrical Algebraic Decomposition via Triangular Decomposition". In: ISSAC. 2009, pp. 95–102. doi: 10.1145/1576702.1576718.
- Alessandro Cimatti, Alberto Griggio, Ahmed Irfan, Marco Roveri, and Roberto Sebastiani. "Incremental Linearization for Satisfiability and Verification Modulo Nonlinear Arithmetic and Transcendental Functions". In: ACM Transactions on Computational Logic 19 (3 2018), 19:1–19:52. doi: 10.1145/3230639.
- George E. Collins. "Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition–Preliminary Report". In: ACM SIGSAM Bulletin 8 (3 1974), pp. 80–90. doi: 10.1145/1086837.1086852.
- George E. Collins. "Quantifier Elimination by Cylindrical Algebraic Decomposition — Twenty Years of Progress". In: Quantifier Elimination and Cylindrical Algebraic Decomposition. 1998, pp. 8–23. doi: 10.1007/978-3-7091-9459-1_2.
- Andreas Dolzmann and Thomas Sturm. "REDLOG: Computer Algebra Meets Computer Logic". In: ACM SIGSAM Bulletin 31 (2 1997), pp. 2–9. doi: 10.1145/261320.261324.

References III

- Matthew England, Russell Bradford, and James H. Davenport. "Improving the Use of Equational Constraints in Cylindrical Algebraic Decomposition". In: ISSAC. 2015, pp. 165–172. doi: 10.1145/2755996.2756678.
- Matthew England, Russell Bradford, James H. Davenport, and David Wilson. "Choosing a Variable Ordering for Truth-Table Invariant Cylindrical Algebraic Decomposition by Incremental Triangular Decomposition". In: ICMS. Vol. 8592. 2014. doi: 10.1007/978-3-662-44199-2_68.
- Dorian Florescu and Matthew England. "Algorithmically Generating New Algebraic Features of Polynomial Systems for Machine Learning". In: SC^2. SIAM AG. Vol. 2460. 2019. url: http://ceur-ws.org/Vol-2460/paper4.pdf.
- Pascal Fontaine, Mizuhito Ogawa, Thomas Sturm, Van Khanh To, and Xuan Tung Vu. "Wrapping Computer Algebra is Surprisingly Successful for Non-Linear SMT". In: SC^2. FLoC. Vol. 2189. 2018, pp. 110–117. url: http://ceur-ws.org/Vol-2189/paper3.pdf.
- Pascal Fontaine, Mizuhito Ogawa, Thomas Sturm, and Xuan Tung Vu. "Subtropical Satisfiability". In: FroCoS. 2017, pp. 189–206.
- Sicun Gao, Soonho Kong, and Edmund M. Clarke. "dReal: An SMT Solver for Nonlinear Theories over the Reals". In: CADE-24. Vol. 7898. 2013, pp. 208–214. doi: 10.1007/978-3-642-38574-2_14.
- D. Yu. Grigor'ev and N.N. Vorobjov. "Solving Systems of Polynomial Inequalities in Subexponential Time". In: Journal of Symbolic Computation 5 (1–2 1988), pp. 37–64. doi: 10.1016/S0747-7171(88)80005-1.

References IV

- Rebecca Haehn, Gereon Kremer, and Erika Ábrahám. "Evaluation of Equational Constraints for CAD in SMT Solving". In: SC^2. FLoC. Vol. 2189. 2018, pp. 19–32. url: http://ceur-ws.org/Vol-2189/paper10.pdf.
- Hoon Hong. "An Improvement of the Projection Operator in Cylindrical Algebraic Decomposition". In: ISSAC. 1990, pp. 261–264. doi: 10.1145/96877.96943.
- Hoon Hong. Comparison of Several Decision Algorithms for the Existential Theory of the Reals. Research rep. Johannes Kepler University, 1991, pp. 1–33.
- Zongyan Huang, Matthew England, David Wilson, James H. Davenport, Lawrence C. Paulson, and James Bridge. "Applying Machine Learning to the Problem of Choosing a Heuristic to Select the Variable Ordering for Cylindrical Algebraic Decomposition". In: CICM. 2014, pp. 92–107. doi: 10.1007/978-3-319-08434-3_8.
- Ahmed Irfan. "Incremental Linearization for Satisfiability and Verification Modulo Nonlinear Arithmetic and Transcendental Functions". PhD thesis. University of Trento, 2018. url: http://eprints-phd.biblio.unitn.it/2952/.
- Dejan Jovanović, Clark Barrett, and Leonardo de Moura. "The Design and Implementation of the Model Constructing Satisfiability Calculus". In: FMCAD. 2013, pp. 173–180. doi: 10.1109/FMCAD.2013.7027033.
- Dejan Jovanović and Leonardo de Moura. "Solving Non-linear Arithmetic". In: IJCAR. Vol. 7364. 2012, pp. 339–354. doi: 10.1007/978-3-642-31365-3_27.

References V

- Sebastian Junges. "On Gröbner Bases in SMT-Compliant Decision Procedures". Bachelor's thesis. RWTH Aachen University, 2012.
- Marek Košta. "New Concepts for Real Quantifier Elimination by Virtual Substitution". PhD thesis. Saarland University, Saarbrücken, Germany, 2016. doi: 10.22028/D291-26679.
- Marek Košta and Thomas Sturm. "A Generalized Framework for Virtual Substitution". In: arXiv e-prints (2015). arXiv: 1501.05826.
- Gereon Kremer. "Isolating Real Roots Using Adaptable-Precision Interval Arithmetic". Master's thesis. RWTH Aachen University, 2013.
- Gereon Kremer. "Computer Algebra and Computer Science". In: ACA. Abstract. 2018, p. 27. doi: 10.15304/9788416954872.
- Gereon Kremer. "Cylindrical Algebraic Decomposition for Nonlinear Arithmetic Problems". PhD thesis. RWTH Aachen University, 2020. url: http://aib.informatik.rwth-aachen.de/2020/2020-04.pdf.
- Gereon Kremer, Florian Corzilius, and Erika Ábrahám. "A Generalised Branch-and-Bound Approach and Its Application in SAT Modulo Nonlinear Integer Arithmetic". In: CASC. Vol. 9890. 2016, pp. 315–335. doi: 10.1007/978-3-319-45641-6_21.
- Daniel Lazard. "An Improved Projection for Cylindrical Algebraic Decomposition". In: Algebraic Geometry and its Applications. 1994. Chap. 29, pp. 467–476. doi: 10.1007/978-1-4612-2628-4_29.

References VI

- Ulrich Loup, Karsten Scheibler, Florian Corzilius, Erika Ábrahám, and Bernd Becker. "A Symbiosis of Interval Constraint Propagation and Cylindrical Algebraic Decomposition". In: CADE-24. Vol. 7898. 2013, pp. 193–207. doi: 10.1007/978-3-642-38574-2_13.
- Scott McCallum. "An Improved Projection Operation for Cylindrical Algebraic Decomposition". PhD thesis. University of Wisconsin-Madison, 1984. url: https://research.cs.wisc.edu/techreports/1985/TR578.pdf.
- Scott McCallum. "An Improved Projection Operation for Cylindrical Algebraic Decomposition of Three-dimensional Space". In: Journal of Symbolic Computation 5 (1–2 1988), pp. 141–161. doi: 10.1016/S0747-7171(88)80010-5.
- Scott McCallum. "On Projection in CAD-based Quantifier Elimination with Equational Constraint". In: ISSAC. 1999, pp. 145–149. doi: 10.1145/309831.309892.
- Scott McCallum. "On Propagation of Equational Constraints in CAD-based Quantifier Elimination". In: ISSAC. 2001, pp. 223–231. doi: 10.1145/384101.384132.
- Scott McCallum and Hoon Hong. "On using Lazard's projection in CAD construction". In: Journal of Symbolic Computation 72 (2016), pp. 65–81. doi: 10.1016/j.jsc.2015.02.001.
- Scott McCallum, Adam Parusiński, and Laurentiu Paunescu. "Validity proof of Lazard's method for CAD construction". In: Journal of Symbolic Computation 92 (2019), pp. 52–69. doi: 10.1016/j.jsc.2017.12.002.
- Leonardo de Moura and Dejan Jovanović. "A Model-Constructing Satisfiability Calculus". In: VMCAI. Vol. 7737. 2013, pp. 1–12. doi: 10.1007/978-3-642-35873-9_1.

References VII

- Akshar Nair, James Davenport, and Gregory Sankaran. "On Benefits of Equality Constraints in Lex-Least Invariant CAD". In: SC^2. SIAM AG. Vol. 2460. 2019. url: http://ceur-ws.org/Vol-2460/paper6.pdf.
- Jasper Nalbach. "Embedding the Virtual Substitution in the MCSAT Framework". Bachelor's thesis. RWTH Aachen University, 2017.
- Jasper Nalbach, Gereon Kremer, and Erika Ábrahám. "On Variable Orderings in MCSAT for Non-linear Real Arithmetic (extended abstract)". In: SC^2. SIAM AG. Vol. 2460. 2019. url: http://ceur-ws.org/Vol-2460/paper5.pdf.
- Greg Nelson and Derek C. Oppen. "Simplification by Cooperating Decision Procedures". In: ACM Transactions on Programming Languages and Systems 1 (2 1979), pp. 245–257. doi: 10.1145/357073.357079.
- James Renegar. "A Faster PSPACE Algorithm for Deciding the Existential Theory of the Reals". In: SFCS. 1988, pp. 291–295. doi: 10.1109/SFCS.1988.21945.
- Andrew Reynolds, Haniel Barbosa, Andres Nötzli, Clark Barrett, and Cesare Tinelli. "CVC4SY: smart and fast term enumeration for syntax-guided synthesis". In: Springer. 2019, pp. 74–83.
- Karsten Scheibler, Stefan Kupferschmid, and Bernd Becker. "Recent Improvements in the SMT Solver iSAT". In: MBMV. Vol. 13. 2013, pp. 231–241.
- Stefan Schupp. "Interval Constraint Propagation in SMT Compliant Decision Procedures". Master's thesis. RWTH Aachen University, 2013.

References VIII

- Andreas Seidl and Thomas Sturm. "A Generic Projection Operator for Partial Cylindrical Algebraic Decomposition". In: ISSAC. 2003, pp. 240–247. doi: 10.1145/860854.860903.
- Adam W. Strzeboński. "Solving Systems of Strict Polynomial Inequalities". In: Journal of Symbolic Computation 29 (3 2000), pp. 471–480. doi: 10.1006/jsco.1999.0327.
- Adam W. Strzeboński. "Cylindrical Algebraic Decomposition Using Local Projections". In: ISSAC. 2014, pp. 389–396. doi: 10.1145/2608628.2608633.
- Alfred Tarski. A Decision Method for Elementary Algebra and Geometry. Research rep. RAND Corporation, 1951. url: https://www.rand.org/pubs/reports/R109.html.
- Vu Xuan Tung, To Van Khanh, and Mizuhito Ogawa. "raSAT: an SMT solver for polynomial constraints". In: Formal Methods in System Design 51 (3 2017), pp. 462–499. doi: 10.1007/s10703-017-0284-9.
- Volker Weispfenning. "The Complexity of Linear Problems in Fields". In: Journal of Symbolic Computation 5 (1–2 1988), pp. 3–27. doi: 10.1016/S0747-7171(88)80003-8.
- Volker Weispfenning. "Quantifier Elimination for Real Algebra — the Quadratic Case and Beyond". In: Applicable Algebra in Engineering, Communication and Computing 8 (2 1997), pp. 85–101. doi: 10.1007/s002000050055.