On the Implementation of Cylindrical Algebraic Coverings for Satisfiability Modulo Theories Solving

Gereon Kremer, Erika Ábrahám, Matthew England and James H. Davenport

Cylindrical Algebraic Coverings in a nutshell

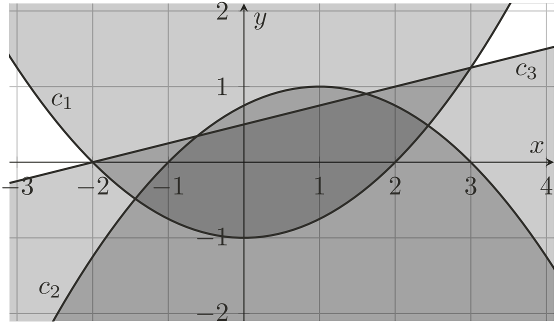
- Fix a variable ordering
- For the kth variable
  - Use constraints to exclude unsatisfiable intervals
  - Guess a value for the kth variable
  - Recurse to $k+1$st variable and obtain
    - a full variable assignment ($\rightarrow$ return SAT)
    - or a covering for the $k+1$st variable
  - Use CAD machinery to infer an interval from this covering
- Until the collected intervals form a covering for the kth variable

Called for the first variable, we get either

- a model, or
- a conflict (formulated as a covering).

An example

$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$



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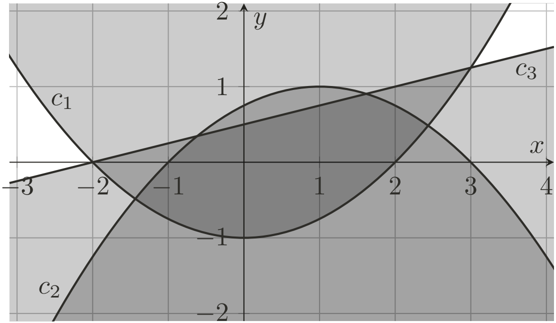
No constraint for x



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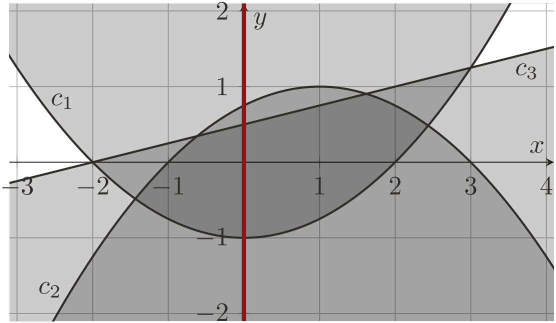

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Guess $x \mapsto 0$

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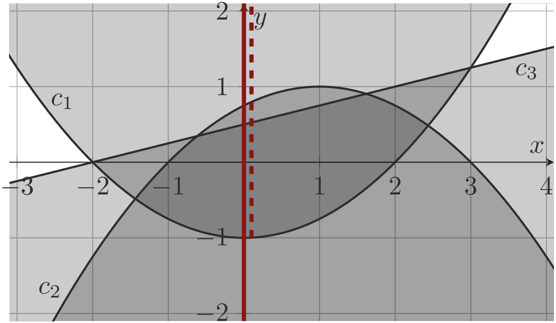

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$c_1 \to y \notin (-1, \infty)$

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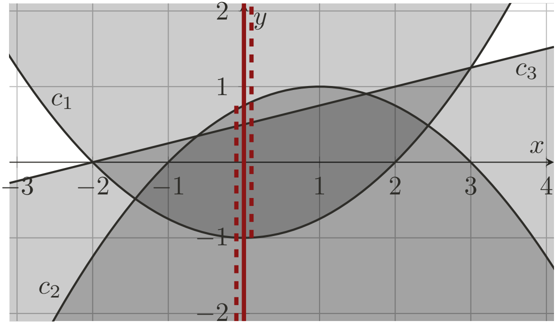

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$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

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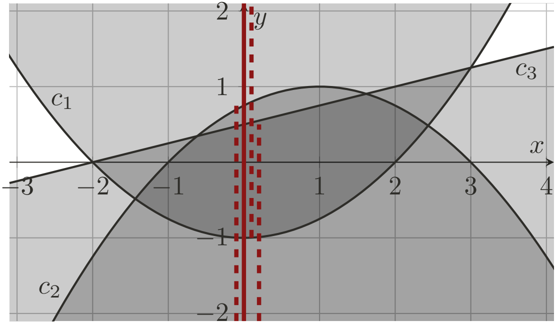

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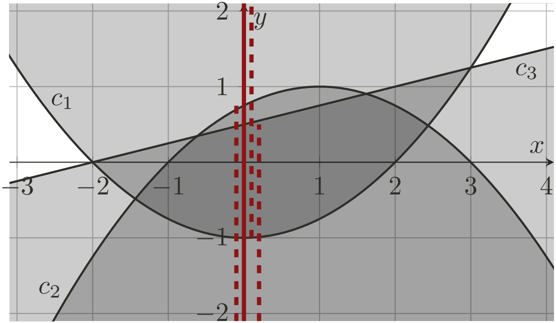

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Construct covering

$(-\infty, 0.5), (-1, \infty)$

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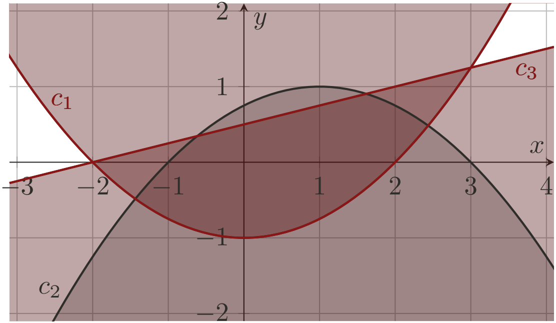

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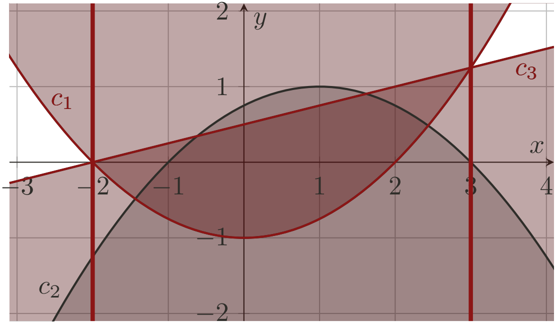

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Construct interval for x

$x \notin (-2, 3)$

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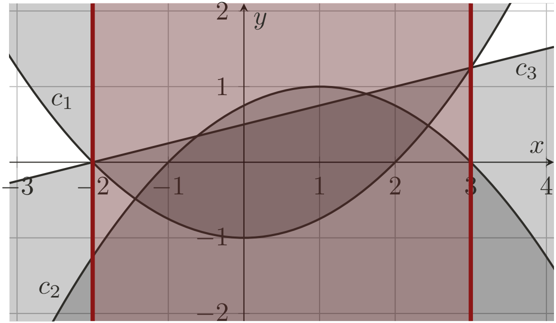

No constraint for x
Guess $x \mapsto 0$
$c_1 \to y \notin (-1, \infty)$
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Construct covering
$(-\infty, 0.5), (-1, \infty)$
Construct interval for x
$x \notin (-2, 3)$
New guess for x

The main algorithm

```
function get_unsat_cover((s_1, ..., s_{i-1}))
```

$\mathbb{I}$:= `get_unsat_intervals(s)`
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  s_i := `sample_outside`($\mathbb{I}$)
  if $i = n$ **then return** $(\mathrm{SAT}, (s_1, \dots, s_{i-1}, s_i))$
  (f, O) := `get_unsat_cover`($(s_1, \dots, s_{i-1}, s_i)$)
  if $f = \mathrm{SAT}$ **then return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    R := `construct_characterization`($(s_1, \dots, s_{i-1}, s_i), O$)
    J := `interval_from_characterization`($(s_1, \dots, s_{i-1}), s_i, R$)
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
return $(\mathrm{UNSAT}, \mathbb{I})$

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function get_unsat_cover((s_1, ..., s_{i-1}))

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Real root isolation over a partial sample point

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Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus I$

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Recurse to next variable

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CAD-style projection: Roots of polynomials restrict where covering is still applicable

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Real root isolation over a partial sample point

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CAD-style projection: Roots of polynomials restrict where covering is still applicable

Extract interval from polynomials

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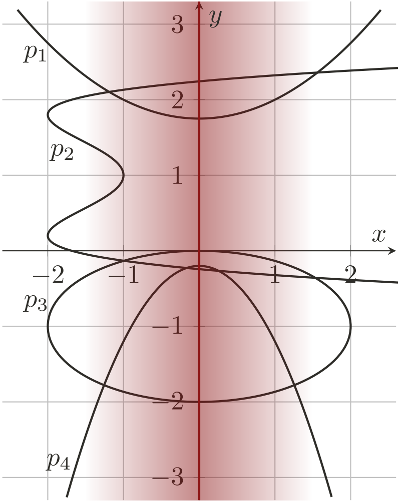
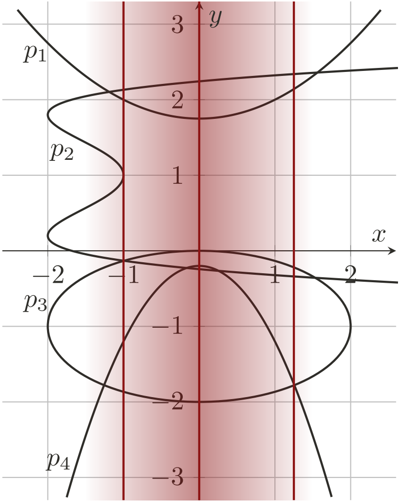


Identify region around sample

construct_characterization



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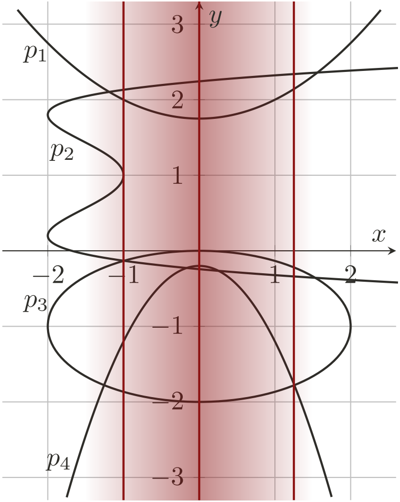
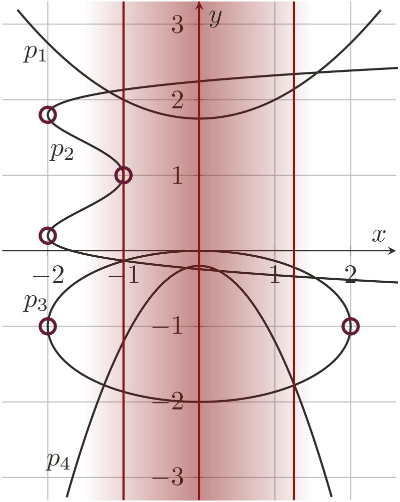


Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

construct_characterization



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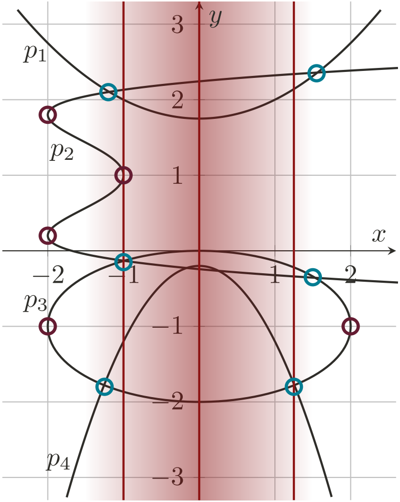

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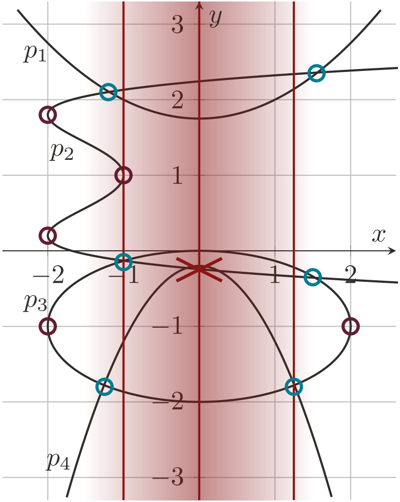

Identify region around sample

CAD projection:

- Discriminants (and coefficients)
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Improvement over CAD:

- Resultants between neighbouring intervals only!

On the implementation in cvc5

- Heavily based on LibPoly [Jovanovic et al. 2017]
- Implements Lazard's lifting [Lazard 1994] using CoCoALib [Abbott et al. 2018]
- Different variable orderings based on [England et al. 2014]
  Nothing spectacular, though
- Generates infeasible subsets
- Allows for partial checks
  Not useful in our context as lemmas are nonlinear
- Supports mixed-integer problems
- Experimental support for incremental checks
  No performance benefit observed

Courtesy of cvc5

- Integrated with linear solver [Cimatti et al. 2018]
  Incremental linearization scheme
- Generates formal proofs
  Not detailed enough yet for automated verification
- Arbitrary theory combination
  It just works!
- Applicable to quantified problems
  No changes necessary!

Experiments

First implemented in SMT-RAT

- Preliminary implementation (no incrementality, no optimizations)
- Easily outperforms regular CAD (from [Kremer et al. 2020])

Second implementation in cvc5: winner of QF_NRA @ SMT-COMP 2021

Solver	SAT	UNSAT	overall	
cvc5	5021	5377	10398	90.0%
yices (NLSAT)	4904	5437	10341	89.5%
z3 (NLSAT)	5093	5195	10288	89.1%
SMT-RAT	4438	4435	8873	76.8%
cvc5 (without CAC)	3283	5385	8668	75.0%
cvc5 (no nl reasoning)	2203	3271	5474	47.4%

References I

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