Cooperating Techniques for Solving Nonlinear Real Arithmetic in the cvc5 SMT Solver

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Satisfiability Modulo Theories

$$\exists \overline{x}. \varphi(\overline{x})$$

Is an existential first-order formula satisfiable?

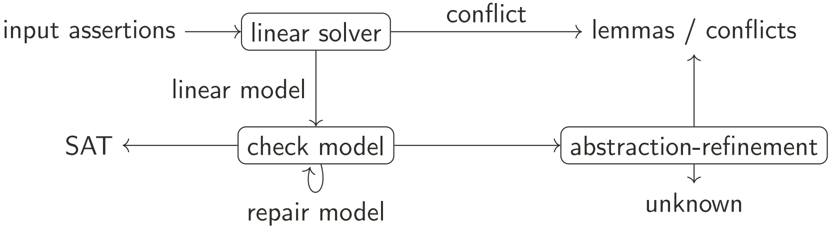
Theories:

- uninterpreted functions
- arrays
- bit-vectors
- floating-point numbers
- arithmetic
- datatypes
- strings
- ...

Extensions:

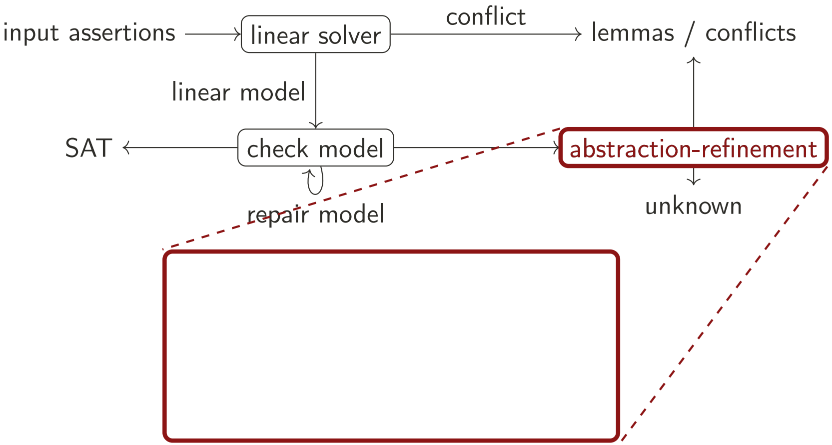
- model generation
- unsat cores
- quantifiers
- optimization queries
- interpolants
- formal proofs
- ...

Arithmetic solving in cvc5



based on [Cimatti et al. 2018]

Arithmetic solving in cvc5



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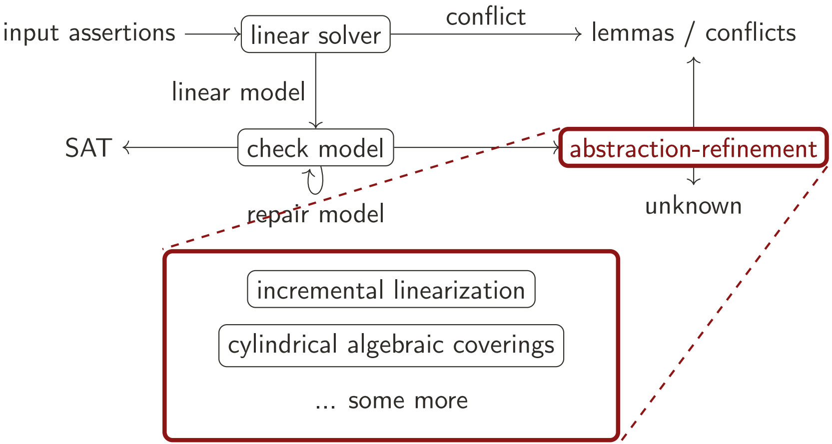
Arithmetic solving in cvc5



based on [Cimatti et al. 2018]

Incremental linearization

implicitly linearize: $x \cdot y \rightsquigarrow a_{x \cdot y}$

$$x > 2 \land y > -1 \land x \cdot y < 2$$

[Cimatti et al. 2018]

Incremental linearization

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$$\begin{aligned} & x > 2 \wedge y > -1 \wedge x \cdot y < 2 \\ \text{Model: } & x \mapsto 3, y \mapsto 0, x \cdot y \mapsto 1 \\ \text{Lemma: } & y = 0 \Rightarrow x \cdot y = 0 \end{aligned}$$

[Cimatti et al. 2018]

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\text{Model: } & x \mapsto 3, y \mapsto 1, x \cdot y \mapsto 0 \\
\text{Lemma: } & (x = 3 \land y = 1) \Rightarrow x \cdot y = 3
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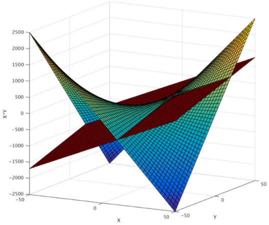

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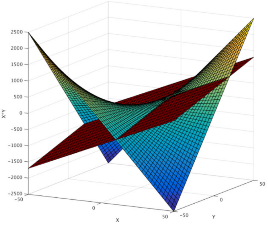
\text{Lemma:}\ & \cancel{(x = 3 \land y = 1) \Rightarrow x \cdot y = 3} \\
& (x \leq 3 \land y \leq 1) \lor (x \geq 3 \land y \geq 1) \\
\Leftrightarrow & (x \cdot y \geq 1 \cdot x + 3 \cdot y - 3 \cdot 1)
\end{aligned}
$$



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Incremental linearization – schemas

split zero	$\top \Rightarrow (t = 0 \lor t \neq 0)$
sign	$x > 0 \land y > 0 \Rightarrow xy > 0$
	$x = 0 \Rightarrow xyz = 0$
magnitude	$\lvert x\rvert > \lvert y\rvert \Rightarrow \lvert xz\rvert > \lvert yz\rvert$
	$\lvert z\rvert > \lvert y\rvert \land \lvert u\rvert > \lvert w\rvert \land \lvert x\rvert \geq 1 \Rightarrow \lvert zuxx\rvert > \lvert yw\rvert$
bounds	$x > 0 \land y > z + w \Rightarrow xy > x(z + w)$
resolution bounds	$y \geq 0 \land s \leq xz \land xy \leq t \Rightarrow ys \leq zt$
tangent plane	$(x \leq 3 \land y \leq 1) \lor (x \geq 3 \land y \geq 1) \Rightarrow xy \geq x + 3y - 3$

Cylindrical Algebraic Coverings

- Guess partial assignment

$$s_1 \times \cdots \times s_k \times s_{k+1}$$

[Ábrahám et al. 2021] [Kremer et al. 2021]

Cylindrical Algebraic Coverings

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$$s \notin s_1 \times \cdots \times s_k \times (a, b)$$

- Project covering to lower dimension

$$s_1 \times \cdots \times s_k \times \{(-\infty, a), [a, b], \ldots (z, \infty)\} \rightarrow s_1 \times \cdots \times s_{k-1} \times (\alpha, \beta)$$

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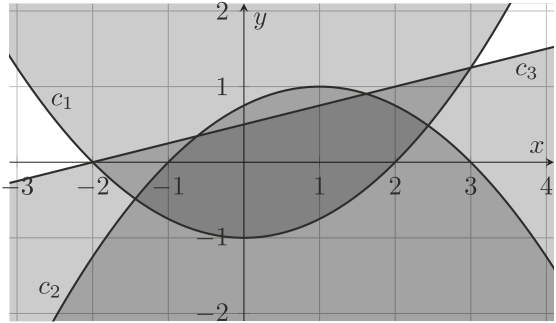
- Eventually get satisfying assignment or a covering in first dimension

$$s = s_1 \times \cdots \times s_n \qquad \text{or} \qquad s_1 \notin \{(-\infty, a), [a, b], \ldots (z, \infty)\}$$

[Ábrahám et al. 2021] [Kremer et al. 2021]

Cylindrical Algebraic Coverings – example

$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$



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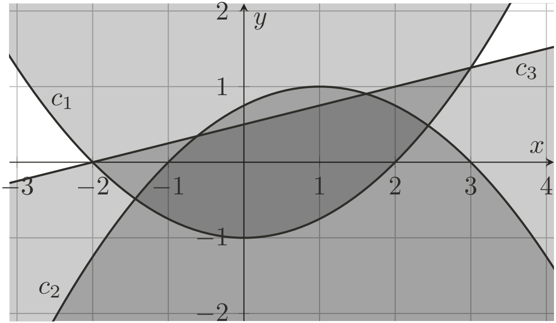

No constraint for x

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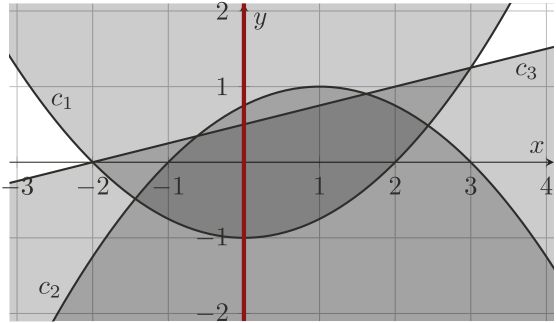

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Guess $x \mapsto 0$

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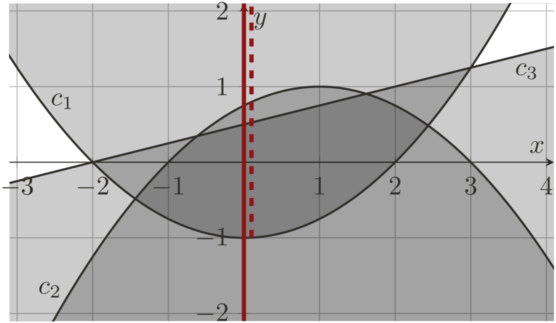

No constraint for x
Guess $x \mapsto 0$
$c_1 \rightarrow y \notin (-1, \infty)$

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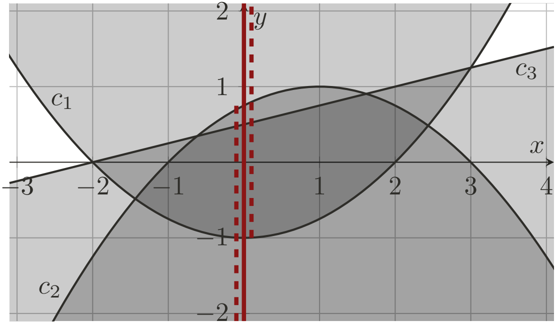

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Guess $x \mapsto 0$

$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

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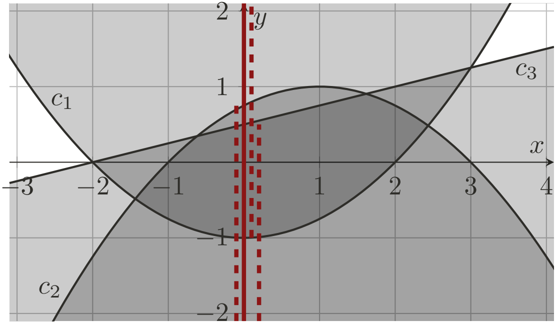

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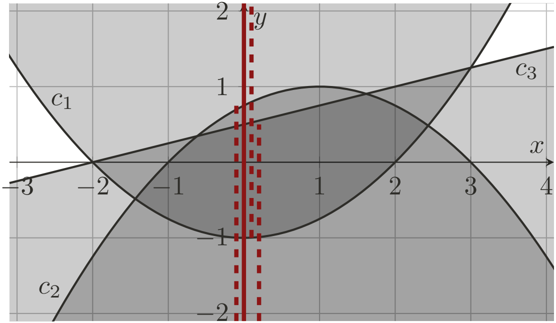

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Construct covering
$(-\infty, 0.5), (-1, \infty)$

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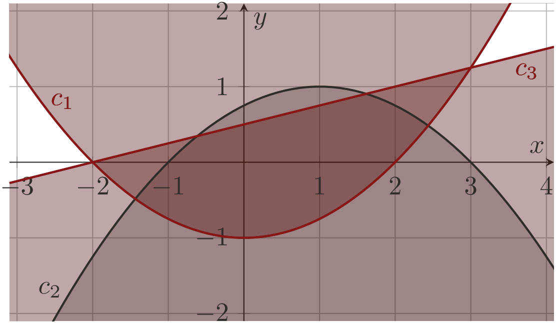

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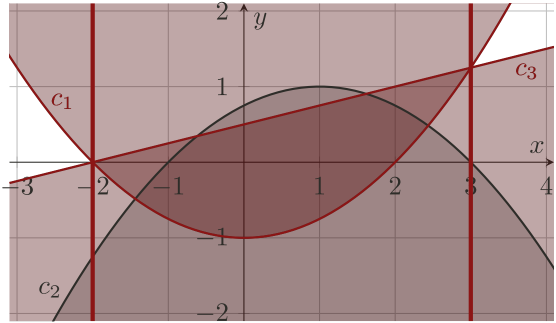

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Construct covering

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Construct interval for x

$x \notin (-2, 3)$

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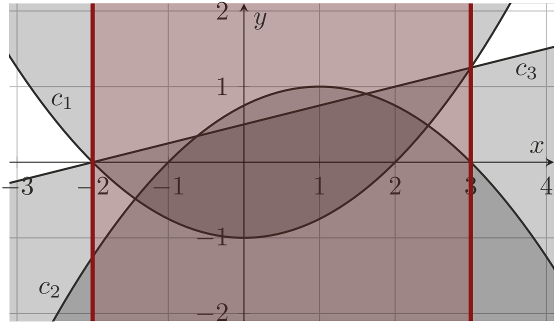

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Construct covering

$(-\infty, 0.5), (-1, \infty)$

Construct interval for x

$x \notin (-2, 3)$

New guess for x

Cylindrical Algebraic Coverings – implementation

- Heavily based on LibPoly [Jovanovic et al. 2017]
- Implements stuff beyond [Ábrahám et al. 2021]:
  - Different projection operators (McCallum, Lazard)
  - Lazard's lifting [Lazard 1994] [Kremer et al. 2021] using CoCoALib [Abbott et al. 2018]
  - Different variable orderings inspired by [England et al. 2014]
  - Generates infeasible subsets
    Store assertions with every interval
  - Supports mixed-integer problems using naive B&B-style intervals
  - Generation of formal proof skeletons
    Helps understanding, not detailed enough for automated verification
  - Experimental support for incremental checks
    No performance benefit observed, lives in a branch
- Arbitrary theory combination
  Real algebraic numbers are first-class citizens of cvc5

Experiments

	QF_NRA	sat	unsat	solved
→	cvc5	**5137**	**5596**	**10733**
	Yices2 2.6.4	4966	5450	10416
	z3 4.8.14	5136	5207	10343
→	cvc5.cov	5001	5077	10078
	SMT-RAT 19.10.560	4828	5038	9866
	veriT+raSAT+Redlog	4522	5034	9556
	MathSAT 5.6.6	3645	5357	9002
→	cvc5.inclin	3421	5376	8797

Abstraction-refinement – extensions

Supports extended operators using incremental linearization:

- transcendentals ($\pi, \sin, \cos, \tan, \ldots$)
- exponentials ($\exp$)
- bitwise `and` on integers (`IAND`, `bvand` in arithmetic)
- power of two (`POW2`, bit shift in arithmetic)

Easily integrates other solving techniques:

- sub-solver should
  - generate a (preferably) linear lemma that rejects the current model
  - find a proper model
- implemented: ICP-style propagations
- ideas: VTS, GB-style conflicts, subtropical satisfiability, ...

Conclusion

- combines linearization and coverings
- conceptually simple strategy
- easily integrates other techniques
- there is more to do...

Any questions?

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