Cylindrical Algebraic Coverings for Quantifiers

Gereon Kremer, Jasper Nalbach

August 12, 2022
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Familiar with Cylindrical Algebraic Coverings?
This works just as you would expect.
Also: no implementation or experiments
Thanks to Dagstuhl Seminar 22072

Disclaimer & Acknowledgements
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The Circle of Life – NRA edition
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Cylindrical Algebraic Coverings for Quantifiers

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The Circle of Life – NRA edition
Guess partial assignment

\[ s_1 \times \cdots \times s_k \times s_{k+1} \]
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Refute partial assignment using intervals

\[ s \notin s_1 \times \cdots \times s_k \times (a, b) \]
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\textbf{Refute partial assignment using intervals}

\[ s \notin s_1 \times \cdots \times s_k \times (a, b) \]

\textbf{Lift covering to lower dimension}

\[ s_1 \times \cdots \times s_k \times \{(-\infty, a), [a, b], \ldots (z, \infty)\} \rightarrow s_1 \times \cdots \times s_{k-1} \times (\alpha, \beta) \]
Guess partial assignment

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Refute partial assignment using intervals

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Lift covering to lower dimension

\[ s_1 \times \cdots \times s_k \times \{(-\infty, a), [a, b], \ldots (z, \infty)\} \rightarrow s_1 \times \cdots \times s_{k-1} \times (\alpha, \beta) \]

Eventually get satisfying assignment or a covering in first dimension

\[ s = s_1 \times \cdots \times s_n \quad \text{or} \quad s_1 \not\in \{(-\infty, a), [a, b], \ldots (z, \infty)\} \]

[Ábrahám et al. 2021] [Kremer et al. 2021]
\[ c_1 : 4 \cdot y < x^2 - 4 \quad c_2 : 4 \cdot y > 4 - (x - 1)^2 \quad c_3 : 4 \cdot y > x + 2 \]
Cylindrical Algebraic Coverings – example

\[ c_1 : 4 \cdot y < x^2 - 4 \quad c_2 : 4 \cdot y > 4 - (x - 1)^2 \quad c_3 : 4 \cdot y > x + 2 \]

No constraint for \( x \)
Cylindrical Algebraic Coverings – example

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Guess \( x \mapsto 0 \)
\[ c_1 : 4 \cdot y < x^2 - 4 \quad \quad c_2 : 4 \cdot y > 4 - (x - 1)^2 \quad \quad c_3 : 4 \cdot y > x + 2 \]

No constraint for \( x \)

Guess \( x \mapsto 0 \)

\[ c_1 \rightarrow y \notin (-1, \infty) \]
\[ c_1 : 4 \cdot y < x^2 - 4 \quad c_2 : 4 \cdot y > 4 - (x - 1)^2 \quad c_3 : 4 \cdot y > x + 2 \]

No constraint for \( x \)

Guess \( x \mapsto 0 \)

\( c_1 \mapsto y \notin (-1, \infty) \)

\( c_2 \mapsto y \notin (-\infty, 0.75) \)
\[ c_1 : 4 \cdot y < x^2 - 4 \quad c_2 : 4 \cdot y > 4 - (x - 1)^2 \quad c_3 : 4 \cdot y > x + 2 \]

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Cylindrical Algebraic Coverings – example

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Construct covering \((-\infty, 0.5), (-1, \infty)\)
\[ c_1 : 4 \cdot y < x^2 - 4 \quad c_2 : 4 \cdot y > 4 - (x - 1)^2 \quad c_3 : 4 \cdot y > x + 2 \]

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Construct covering
\[ (-\infty, 0.5), (-1, \infty) \]

Construct interval for \( x \)
\[ x \not\in (-2, 3) \]
\[ c_1 : 4 \cdot y < x^2 - 4 \quad c_2 : 4 \cdot y > 4 - (x - 1)^2 \quad c_3 : 4 \cdot y > x + 2 \]

No constraint for \( x \)

Guess \( x \mapsto 0 \)

\( c_1 \rightarrow y \notin (-1, \infty) \)
\( c_2 \rightarrow y \notin (-\infty, 0.75) \)
\( c_3 \rightarrow y \notin (-\infty, 0.5) \)

Construct covering \((-\infty, 0.5), (-1, \infty)\)

Construct interval for \( x \)

\( x \notin (-2, 3) \)

New guess for \( x \)
We want to characterize both true and false regions of quantified formulae.
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**Core change**

Instead of a model return *satisfying interval with suitable characterization.*
We want to characterize both true and false regions of quantified formulae.

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Challenges:

- **Boolean structure?**
  → consider constraints of (suitable) implicants

- **Model construction?**
  → reconstruct from characterization of true region.

- **Interval in dimension zero?**
  → just a technicality, use $\top, \bot$

- **Termination guarantees?**
  → still the same
∀x. ∃y. y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2
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forall(s = ())
\( \forall x. \exists y. \quad y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2 \)
∀x.∃y. y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2

forall(s = ())
exists(s = (2))
SAT @ s = (2, 3.5)
\[ \forall x. \exists y. y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2 \]

forall(s = ( ))
exists(s = (2))
SAT @ s = (2, 3.5)
SAT @ s \times (1, 3)
∀x.∃y. y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2

forall(s = (\))
exists(s = (2))
SAT @ s = (2, 3.5)
SAT @ s \times (1, 3)
exists(s = (4))
∀x.∃y. y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2

forall(s = ())
exists(s = (2))
SAT @ s = (2, 3.5)
SAT @ s \times (1, 3)
exists(s = (4))
UNSAT @ s \times (3, \infty)
UNSAT @ s \times (-\infty, 3.5)
\[ \forall x. \exists y. \quad y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2 \]

forall(s = ())
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\[ \forall x. \exists y. \quad y > 3.5 - 2(x - 4)^2 \quad (x - 2)^2 + (y - 2)^2 - 1 > 0 \quad y < 3 + 0.25(x - 4)^2 \]

\[
\begin{align*}
&\text{forall}(s = ()) \\
&\text{exists}(s = (2)) \\
&\quad \text{SAT @ } s = (2, 3.5) \\
&\quad \text{SAT @ } s \times (1, 3) \\
&\quad \text{exists}(s = (4)) \\
&\quad \text{UNSAT @ } s \times (3, \infty) \\
&\quad \text{UNSAT @ } s \times (-\infty, 3.5) \\
&\quad \text{UNSAT @ } s \times (3.5, 4.5) \\
&\text{UNSAT}
\end{align*}
\]
Consider free variables first

Use previous approach for bounded variables

Collect all SAT regions for free variables

Use solution formula construction from [Brown 1999]

Return disjunction of all SAT regions
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Conclusion

- Cylindrical Algebraic Coverings can be adapted to quantifiers and QE
- It mostly works as you would think it does
- A number of subtle challenges
- Provides for a few nice generalizations
- No implementation yet
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Thank you for your attention!
Any questions?
References

