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Linearization Techniques for Nonlinear Arithmetic Problems in SMT

Linearisierungstechniken für nichtlineare, arithmetische Probleme in SMT

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Abstract

Polynomial constraint solving plays a prominent role in several areas of soft- and hardware verification, optimization and planning. Unfortunately, the nonlinear constraint solving problem over the integers is undecidable. The situation is not much better when considering the reals since, although the problem is decidable as it was shown for the first-order theory of real closed fields by Tarski, using the related algorithms in practice is unfeasible due to their complexity. More efficient, but incomplete decision procedures implementing sufficient conditions only are hence applied to decide simpler instances prior to a call of more elaborate decision procedures. In this thesis we present the theoretical foundations of two new incomplete modules for the Satisfiability Modulo Theories (SMT) toolbox `SMT-RAT`, namely the subtropical and the case-splitting methods. They are dedicated to proving satisfiability of nonlinear real and integer arithmetic formulas by encoding them into an SMT problem considering only linear arithmetic. These linearizations are in turn solved using linear arithmetic solvers implementing a Simplex or Branch-and-Bound approach, respectively. Extensive experiments on the `SMT-LIB` benchmarks demonstrate that these methods are not strong decision procedures by themselves but valuable heuristics to use within a portfolio of techniques.

Contents

Introduction

The *satisfiability problem* poses the question of whether there exists an assignment to the variables of a given logical formula such that the later becomes true. Propositional logic is well-suited for a broad range of problems like the verification of logic programs or the bounded model checking of discrete systems. Accordingly, a lot of effort has been put into the development of fast solvers for the propositional satisfiability problem (SAT). Other inherently continuous problems in the areas of system analysis and verification require the expressiveness of theories. Therefore, propositional logic is extended with first-order theory constraints to so called *Satisfiability Modulo Theories* (SMT).

Especially polynomial constraints are ubiquitous and it is paramount to have efficient automatic tools that, given a polynomial constraint with integer or real indeterminates, either return a solution or notify that the constraint is unsatisfiable. Unfortunately, the polynomial constraint solving problem over the integers is undecidable. The situation is not much better when considering the reals since, although the problem is decidable as it was shown for the first-order theory of real closed fields by Tarski, using the related algorithms in practice is unfeasible due to their complexity. Therefore, all methods used in practice for both integer and real solution domains are incomplete. There are two approaches, namely focusing on proving satisfiability or focusing on proving unsatisfiability. In general, the decision on the approach is guided by the problem in hand. Current techniques focusing on satisfiability encode the problem into SAT known as bit-blasting. Following the success of the translation into SAT, it is reasonable to consider whether there is a better target language than propositional logic to keep as much as possible the arithmetic structure of the source language. Thus, in this thesis we consider methods for solving nonlinear constraints based on encoding the problem into an SMT problem over linear real or integer arithmetic. An interesting feature of this approach is that, in contrast to SAT translations, by having linear arithmetic built into the language, negative values and sums can be handled without additional codification effort.

Chapter 1.

Preliminaries

In this chapter we give an overview of the classical approaches in SMT solving both in general and in the context of the SMT-RAT framework. For this purpose, we first define the notion of SMT problems for arbitrary underlying theories and introduce the existential fragments of nonlinear real and integer arithmetics as two of their most important instances. We then sketch the basic scheme of DPLL-based SMT solving and show its impact on the modular design of the SMT-RAT framework. This will clarify the setting in which our own implementationary work is settled.

1.1. Satisfiability Modulo Theories

The Satisfiability Modulo Theories (SMT) problem is a generalization of the well-known satisfiability (SAT) problem. A SAT problem instance consists of a formula Φ in propositional logic, which is a combination of Boolean-valued variables $b_1, \ldots, b_m$ with connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$. It asks for an interpretation $\mathcal{I} : \{b_1, \ldots, b_m\} \rightarrow \{\texttt{True}, \texttt{False}\}$ of the variables, such that Φ evaluates under $\mathcal{I}$ to True, which is abbreviated by $\mathcal{I} \models \Phi$.

The SMT problem replaces the Boolean-valued variables in favor of *constraints* $c_1, \ldots, c_m$ that are expressed in the context of a *theory* $\mathcal{T}$, consisting of a *domain* $\mathcal{D}$ (like $\mathbb{R}$) alongside with interpretations for all *function symbols* $f_1, \ldots, f_k$ (like $+$) and *predicate symbols* $\sim_1, \ldots, \sim_l$ (like $<$). The variables $x_1, \ldots, x_d$ in Φ are no longer Boolean-valued, but range over the domain $\mathcal{D}$. Solving the SMT instance Φ in the theory $\mathcal{T}$ means deciding whether an interpretation $\mathcal{I} : \{x_1, \ldots, x_d\} \rightarrow \mathcal{D}$ exists, such that $\mathcal{I} \models \Phi$ with respect to $\mathcal{T}$.

The SAT problem is known to be NP-hard, though decidable, since we may simply enumerate all possible interpretations for a given instance. The decidability of the SMT problem, on the other hand, depends heavily on the underlying theory $\mathcal{T}$ and additional restriction to specific fragments of the first-order logic. The focus of this thesis lies the quantifier-free fragment of the nonlinear real and integer arithmetic:

Definition 1.1 The syntax of a formula in the quantifier-free fragment of the nonlinear real and integer arithmetic is defined by the following grammar:

$$
\begin{aligned}
\mathit{formula} &::= \mathit{constraint} \mid (\neg \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\mathit{formula} \vee \mathit{formula}) \\
\mathit{constraint} &::= \mathit{term} \sim \mathit{term} \qquad \text{for} \qquad \sim \in \{<\leq, =, \neq, \geq, >\} \\
\mathit{term} &::= v \mid c \mid \mathit{term} + \mathit{term} \mid \mathit{term} \cdot \mathit{term} \qquad \text{for} \qquad v \in \{x_1, \dots, x_d\},\ c \in \mathbb{R}
\end{aligned}
$$

Depending on the domain $\mathcal{D} = \mathbb{R}$ or $\mathcal{D} = \mathbb{Z}$, we distinguish the nonlinear real (QF_NRA) from the integer (QF_NIA) arithmetic and simply write QF_NA do denote any of these nonlinear arithmetic problems.

The QF_NA formulas Φ are hence arbitrarily shaped Boolean combinations of polynomial inequalities. For a concise description of these formulas, we will subsequently rely on the outcome of the the following lightweight normalization steps:

(i) For an exponent vector $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{R}^d$ and a vector of real- or integer-valued variables $\mathbf{x} = (x_1, \dots, x_d)$, we denote by $\mathbf{x} \cdot \mathbf{p} := \sum_{i=1}^{d} x_i p_i$ the usual dot product and by $\mathbf{x}^{\mathbf{p}} := \prod_{i=1}^{d} x_i^{p_i}$ the monomial exponent. Every multivariate polynomial $\mathbf{f}(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_d]$ can now be written in a *sparse distributive notation* as

$$\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} f_{\mathbf{p}} \mathbf{x}^{\mathbf{p}} \qquad \text{with} \qquad \mathrm{fr}(\mathbf{f}) := \{\mathbf{p} \in \mathbb{Z}^d \mid f_{\mathbf{p}} \neq 0\},$$

where the *frame* $\mathrm{fr}(\mathbf{f})$ denotes its supporting set. Every constraint c_i in Φ can thus be written as $\mathbf{f}_i(\mathbf{x}) \sim_i 0$ for a relation symbol $\sim_i \in \{<, \leq, =, \neq, \geq, >\}$.

(ii) The given QF_NA formula Φ can be transformed in linear time into an equisatisfiable formula $\Phi_{\mathtt{CNF}}$ in conjunctive normal form by using Tseitin's encoding to get

$$\Phi_{\mathtt{CNF}} = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{l_i} \ell_{ij} \qquad \text{with} \qquad \ell_{ij} \in \{c_{ij}, \neg c_{ij}\}$$

for QF_NA constraints c_{ij}. Next, the negations in negative literals $\ell_{ij} = \neg c_{ij}$ with a constraint $c_{ij} = \mathbf{f}_{i,j}(\mathbf{x}) \sim_{ij} 0$ can be eliminated by pushing them to the theory constraints: $<$, $\leq$ and $=$ get replaced with $\geq$, $>$ and $\neq$ and vice versa.

In the following, we will assume that these transformations were already done and that Φ is a CNF formula consisting of unnegated constraints in sparse distributive representation

$$\Phi = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{l_i} c_{ij} \qquad \text{with} \qquad c_{ij} = \mathbf{f}_{ij}(\mathbf{x}) \sim_{ij} 0.$$

For $i = 1, 2$ let $\Phi_i := \bigwedge_{j=1}^{k_i} \omega_{i,j}$ be two of these CNF formulas with clauses $\omega_{i,j}$. For a clause ω we denote by $\omega \in \Phi_i$ the existence of an index $j \in \{1, \dots, k_i\}$ with $\omega = \omega_{i,j}$. Following this set theoretic nomenclature, we further write

- $\Phi_1 \subseteq \Phi_2$, if and only if for all $\omega \in \Phi_1$ it holds that $\omega \in \Phi_2$.
- $\Phi_1 \cap \Phi_2$ for a CNF formula with $\omega \in \Phi_1 \cap \Phi_2$ if and only if $\omega \in \Phi_1$ and $\omega \in \Phi_2$.

Any unsatisfiable formula Φ_1 with $\Phi_1 \subseteq \Phi_2$ is called an *unsatisfiable core* of Φ_2.

1.2. DPLL-based SMT solving

Several tools for deciding the satisfiability of SMT formulas over a quantifier-free first-order theory $\mathcal{T}$ rely on the DPLL($\mathcal{T}$) framework: They combine a Boolean satisfiability solver based on the *Davis-Putnam-Logemann-Loveland* (DPLL) procedure to resolve the Boolean structure of a given formula, and a dedicated theory solver capable of verifying the consistency of theory constraints conjunctions. In what follows, we have a closer look on the DPLL(QF_NA) approach (see [KS08, Chapter 2]).

From a normalized input formula Φ as described in the last subsection, the *Boolean abstraction* $\Phi_{\texttt{Bool}}$ is constructed by introducing a fresh Boolean variable e_{ij} for every constraint c_{ij} and keeping the Boolean skeleton intact, which gives

$$\Phi_{\texttt{Bool}} = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{l_i} e_{ij}.$$

A DPLL-based SAT solver operating in less lazy mode now systematically tries to find partial interpretations $\mathcal{I}$ for the variables e_{ij} that do not contradict the Boolean skeleton $\Phi_{\texttt{Bool}}$ (see Figure 1.1). After each variable assignment, the *constraints conjunction*

$$\Phi_{\texttt{Theory}} = \bigwedge_{\mathcal{I} \models e_{ij}} c_{ij}$$

is handed over to the theory solver and checked for consistency. Recall that the Boolean abstraction $\Phi_{\texttt{Bool}}$ does not contain any negations and is therefore a monotone formula. Hence, the original formula Φ must be satisfiable if and only if $\Phi_{\texttt{Theory}}$ is consistent. If the theory solver fails to find a solution of the given constraints conjunction $\Phi_{\texttt{Theory}}$, it provides a preferably minimal unsatisfiable core $\Phi_{\texttt{Inf}} \subseteq \Phi_{\texttt{Theory}}$ as explanation to the SAT solver, which is used to narrow down the search for feasible assignments. Depending on the answer of the theory solver on this constraint conjunction, the SAT solver can adjust its partial solution until a complete assignment is found. The formula is declared to be unsatisfiable, if the SAT solver is not able to find any further interpretations for $\Phi_{\texttt{Bool}}$.

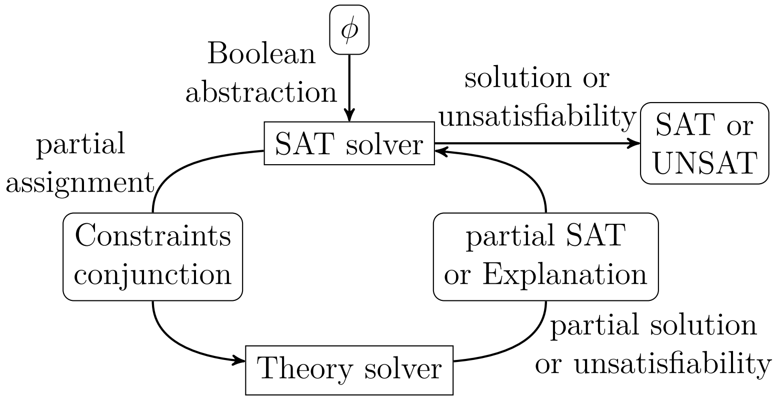


Figure 1.1.: The basic scheme of DPLL-based SMT solving

This DPLL(QF_NA) approach in less lazy mode requires a theory solver for the QF_NA theory that supports the following minimal functionality known as *SMT-compliance*:

Incrementality It has to manage an internal state to make use of previous consistency checks as the input formulas $\Phi_{\texttt{Theory}}$ do not vary too much between two successive invocations of the theory solver. It should therefore allow the belated assertion of new and removal of already asserted constraints to the constraints conjunction.

Correctness and Termination The consistency check over all asserted constraints must terminate in finite time and return a `SAT`/`UNSAT` answer. In case of satisfiability a model must be constructed and returned. Otherwise, it must provide a preferably minimal unsatisfiable core $\Phi_{\texttt{Inf}}$ as explanation for the unsatisfiability of $\Phi_{\texttt{Theory}}$.

For many theory solvers that only implement a sufficient satisfiability condition like our own linearization approaches, the second point is illusory. We therefore allow a third answer `Unknown` that can be returned, if the consistency of $\Phi_{\texttt{Theory}}$ is undecidable.

1.3. SMT-RAT

The SMT toolbox `SMT-RAT` [Cor+12] is an open-source project written in C++ for SMT solving over several background theories. The toolbox is structured into its basic architectural components called *modules* that provide SMT compliant implementations of decision procedures. Every module maintains a list $\mathbf{C}_{\text{rec}}$ of received formulas whose conjunction needs to be checked for consistency the next time when the `check` method is called. This list can be modified in an incremental fashion with the use of the `assert` and `remove` methods to add new and remove already added formulas. Different modules can be stacked together with the help of *managers* to a complete solving strategy. Every

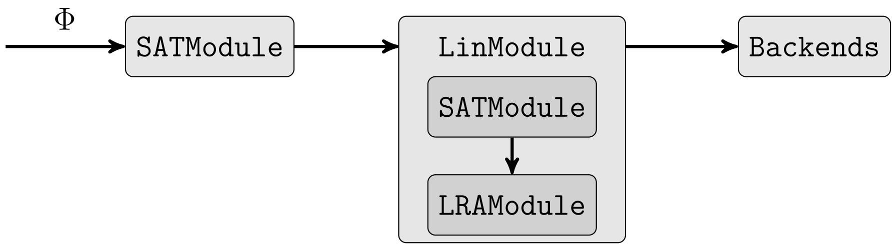


Figure 1.2.: General structure of the strategy tree for our linearization modules.

module decides himself which formulas to delegate to succeeding modules that we call his *backends*. Figure 1.2 shows a strategy in which the prototypical linearization module `LinModule` is a placeholder for any of our two modules `STropModule` and `CSplitModule`. It is preceeded by an instance of a `SATModule` that implements a DPLL-based SAT solver to resolve the Boolean skeleton of the input formula Φ. Our linearization modules therefore do not receive arbitrary formulas, but only conjunctions of polynomial constraints as described in Section 1.2. They perform a linearization of the nonlinear input and pass the result to an internal linear arithmetic solver `LRAModule`. Since the later also expects its own input to be a constraints conjunction, the linearization must be piped beforehand into a second instance of a `SATModule`. The linearization modules are both sound, but incomplete, for which reason they call their `Backends` on their complete input $\mathbf{C}_{\text{rec}}$ in case they are unable to decide the consistency themselves. In our experiments, the `Backend` strategy is a combination of the following decision procedures already implemented as `SMT-RAT` modules:

`LRAModule` This is a misnomer, since it not only implements the Simplex method to tackle linear real arithmetic problems, but also performs Branch-and-Bound on all integer-valued variables to effectively handle any linear mixed-integer problems.

`ICPModule` Interval Constraint Propagation uses the given constraints to iteratively contract the search space until an interval for every variable is reached that tightly over-approximates the solution set satisfying some preset precision requirement.

`VSModule` The Virtual Substitution method exploits the existence of closed form solutions for univariate polynomials up to degree four to successively eliminate variables.

`CADModule` The Cylindrical Algebraic Decomposition algorithm decomposes the search space into a finite number of connected sets called *cells*, on which each polynomial of the input constraints has constant sign. The satisfiability can then be decided by testing their consistency at single sample points in each cell.

Chapter 2.

Subtropical Satisfiability

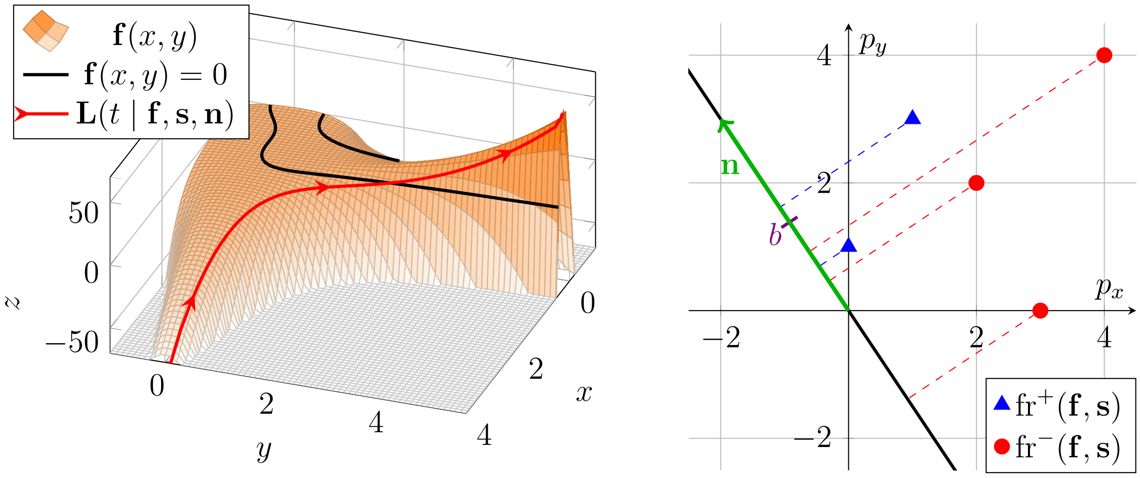
In this chapter we provide the theoretical foundations for our `SMT-RAT` implementation of the subtropical module (`STropModule`) that is fully presented in Appendix A. It is inspired by the incomplete but terminating subtropical real root finding method as described in [Stu15] that identifies roots of very large multivariate polynomials. The algorithm takes an abstract view of a polynomial as the set of its exponent vectors tagged with sign information on the corresponding coefficients. It then examines the limiting behaviour of the polynomial in the direction of a normal vector to find real zeros. The search for such a normal vector is translated into a linear problem in the space of the polynomial's exponent vectors and in turn solved using a linear real arithmetic solver. In the context of nonlinear real arithmetic problems in SMT, this algebraic root finding method was first generalized in [Fon+17] to find solutions of a conjunction of strict polynomial inequalities. In the following sections, we will describe further enhancements that could improve the completeness of the original method with slightly more overhead.

2.1. Limiting behaviour of multivariate polynomials

In the following we will examine sufficient conditions for the existence of solutions to the problem description given below. These results based on the subtropical real root finding method will be used afterwards to decide the satisfiability of a conjunction of nonlinear real and integer arithmetic constraints in the context of Sat Modulo Theories solving.

Problem 2.1 Let $\mathbf{f}(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_d]$ be a multivariate polynomial. Find a real-valued variable assignment $\mathbf{a} \in \mathbb{R}^d$ such that the inequality $\mathbf{f}(\mathbf{a}) > 0$ is fulfilled.

The idea of an incomplete algorithm for Problem 2.1 is best described in the one-dimensional case $d = 1$: For a univariate polynomial $\mathbf{f}(x) \in \mathbb{R}[x]$ consider the limiting process $\lim_{a \to L} \mathbf{f}(a)$, as a approaches one of the one-sided limits $L \in \{\pm 0, \pm\infty\}$. Repeatedly test the sign of $\mathbf{f}(a)$ until finally $\mathbf{f}(a) > 0$ is fulfilled, if this happens at all. The examination of four different limit values L can be reformulated as the choice of a sign $s \in \{\pm 1\}$ and an



(a) Momentum curve $\mathbf{m}(t \mid \mathbf{s}, \mathbf{n}) := (t^{-2}, t^3)$ on its surface that eventually becomes positive.

(b) Projection of frame vertices onto the hyperplane in direction of $\mathbf{n} = (-2, 3)$.

Figure 2.1.: Visualization of the polynomial $\mathbf{f}(x, y) := -4x^4y^4 - x^3 - 3x^2y^2 + 2xy^3 + y$.

exponent $n \in \mathbb{Z}$, and equivalently considering the single limiting process $\lim_{t \to +\infty} \mathbf{f}(st^n)$ instead. The success of this method is clearly predetermined by the leading coefficient of the *Laurent polynomial* expression $\mathbf{L}(t \mid \mathbf{f}, s, n) := \mathbf{f}(st^n) \in \mathbb{R}[t, t^{-1}]$ in a single indeterminate t. The leading coefficient can be calculated prior to the execution of this method to ensure the desired positivity of its sign. In order to generalize this idea to the multivariate case $d > 1$, we similarly parametrize a univariate subcurve of $\mathbf{f}(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_d]$ along which the limiting behaviour of the multivariate polynomial will be explored.

Definition 2.2 Let $\mathbf{s} \in \{\pm 1\}^d$ be a sign vector and let $\mathbf{n} \in \mathbb{Z}^d$ be an exponent vector. The oriented *momentum curve* in the direction of the *normal vector* $\mathbf{n}$ with the *sign variant* $\mathbf{s}$ is given by the mapping $\mathbf{m}(\cdot \mid \mathbf{s}, \mathbf{n}) : \mathbb{R}_+ \to \mathbb{R}^d, t \mapsto (s_1 t^{n_1}, \ldots, s_d t^{n_d})$.

A momentum curve $\mathbf{m}(t \mid \mathbf{s}, \mathbf{n})$ is used to restrict the domain of the polynomial $\mathbf{f}(\mathbf{x})$ via

$$\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n}) := \mathbf{f}(\mathbf{m}(t \mid \mathbf{s}, \mathbf{n})) = \sum_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} f_{\mathbf{p}} (s_1 t^{n_1}, \ldots, s_d t^{n_d})^{\mathbf{p}} = \sum_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}} t^{\mathbf{n} \cdot \mathbf{p}}.$$

The result is a Laurent polynomial $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n}) \in \mathbb{R}[t, t^{-1}]$ in a single indeterminate t that allows an analysis of its limiting behaviour for $t \to +\infty$ just like for ordinary univariate polynomials considered above, provided that its leading coefficient is also positive.

Example 2.3 Consider the bivariate polynomial

$$\mathbf{f}(x, y) := -4x^4y^4 - x^3 - 3x^2y^2 + 2xy^3 + y \in \mathbb{R}[x, y].$$

Figure 2.1a shows an example for a good choice of a normal vector $\mathbf{n} = (-2, 3)$ and a sign variant $\mathbf{s} = (1, 1)$ such that the resulting Laurent polynomial

$$\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n}) = 2t^7 - 4t^4 + t^3 - 3t^2 - t^{-6} \in \mathbb{R}[t, t^{-1}]$$

eventually becomes positive, for instance at $t = 2$. A corresponding satisfying assignment for the inequality $\mathbf{f}(x, y) > 0$ in Problem 2.1 is then given by $\mathbf{a} := \mathbf{m}(t \mid \mathbf{s}, \mathbf{n}) = (2^{-2}, 2^3)$.

The search for a variable assignment $\mathbf{a} \in \mathbb{R}^d$ for the strict inequality $\mathbf{f}(\mathbf{x}) > 0$ hence boils down to a search for a normal vector $\mathbf{n} \in \mathbb{Z}^d$ and a sign variant $\mathbf{s} \in \{\pm 1\}^d$ such that the resulting polynomial $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n})$ has a positive leading coefficient. A corresponding variable assignment for the inequality can then be reconstructed as the image of the momentum curve $\mathbf{m}(t \mid \mathbf{s}, \mathbf{n})$ for a large enough $t \in \mathbb{R}_+$, where $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n}) > 0$ is fulfilled.

2.2. Restriction process as geometric projection

The coefficients of $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n})$ are composed of those of the multivariate polynomial $\mathbf{f}(\mathbf{x})$. To make this calculation process explicit, take the above definition of the Laurent polynomial and reorder its terms according to the same integral exponents to get

$$\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n}) = \sum_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}} t^{\mathbf{n} \cdot \mathbf{p}} = \sum_{k \in \mathbb{Z}} L_k t^k \qquad \text{with} \qquad L_k := \sum_{\substack{\mathbf{p} \in \mathrm{fr}(\mathbf{f}) \\ \mathbf{n} \cdot \mathbf{p} = k}} \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}}.$$

Graphically speaking, the exponent vector $\mathbf{n}$ of the momentum curve defines a hyperplane in the $\mathbb{Z}$-lattice of all possible exponent vectors $\mathbb{Z}^d$ onto which the frame vertices of $\mathbf{f}(\mathbf{x})$ are projected. Those frame vertices $\mathbf{p} \in \mathrm{fr}(\mathbf{f})$ with equal *projection length* $\mathbf{n} \cdot \mathbf{p} = k$ to the origin contribute to the same coefficient L_k weighted with the sign $\mathbf{s}^{\mathbf{p}}$. Let us therefore partition the frame $\mathrm{fr}(\mathbf{f})$ into a *variant positive* and a *variant negative frame* by

$$\mathrm{fr}^+(\mathbf{f}, \mathbf{s}) := \{\mathbf{p} \in \mathrm{fr}(\mathbf{f}) \mid \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}} > 0\} \qquad \text{and} \qquad \mathrm{fr}^-(\mathbf{f}, \mathbf{s}) := \{\mathbf{p} \in \mathrm{fr}(\mathbf{f}) \mid \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}} < 0\}.$$

Example 2.4 To better understand the choice of the exponent vector $\mathbf{n} = (-2, 3)$ in Example 2.3, consider the visualization of the frame $\mathrm{fr}(\mathbf{f})$ in Figure 2.1b and its projection to the hyperplane defined by $\mathbf{n}$. The frame vertex $\mathbf{p} = (1, 3)$ has the largest projection length $\mathbf{n} \cdot \mathbf{p} = 7$ and hence constitutes the leading coefficient $L_7 = \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}} = 2$. Since this is positive, we can deduce the positive divergence of $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n})$ for a large enough $t \in \mathbb{R}_+$.

Let us enumerate all existing projection lengths descendingly by $\mathbf{n} \cdot \mathrm{fr}(\mathbf{f}) = \{k_1, \ldots, k_l\}$ with $k_1 > \ldots > k_l$. As this example suggests, we have a special interest in the sign of the coefficient L_{k_1} with the largest existing projection length $k_1 := \max_{\mathbf{p}\in\mathrm{fr}(\mathbf{f})} \mathbf{n}\cdot\mathbf{p}$. But note that this is not necessarily the leading coefficient of $\mathbf{L}(t \mid \mathbf{f},\mathbf{s},\mathbf{n})$, since the signed coefficients $\mathbf{s}^{\mathbf{p}} f_{\mathbf{p}}$ it is composed of may cancel out each other yielding an overall value of zero. In the worst case, the whole Laurent polynomial can vanish through the projection making it impossible to draw any conclusions about the limiting behaviour of $\mathbf{f}(\mathbf{x})$ for this particular choice of the normal vector $\mathbf{n}$ and the sign variant $\mathbf{s}$. Nonetheless the approaches we will review in the following sections rely on a positivity check of this coefficient with largest projection length only. Linearization based methods that deduce the sign of the true leading coefficient of $\mathbf{L}(t \mid \mathbf{f},\mathbf{s},\mathbf{n})$ performed poorly in all of our experiments.

2.3. Exploiting the linear separabilty of frame vertices

The positivity condition on the coefficient L_{k_1} can be formulated as a linear real arithmetic formula $\Phi_{\mathtt{Constr}}$ without the need of calculating the entire Laurent polynomial $\mathbf{L}(t \mid \mathbf{f},\mathbf{s},\mathbf{n})$. In the following subsections we will derive two variants of this formula $\Phi_{\mathtt{Constr}}$ dedicated to different linearization approaches to test the positivity of $L_{k_1} = \sum_{\mathbf{n}\cdot\mathbf{p}=k_1} \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}}$:

- The linearization $\Phi^{\mathtt{Str}}_{\mathtt{Constr}}$ of Subsection 2.3.1 is based on the original method in [Fon+17]. It verifies whether the coefficient L_{k_1} is composed of variant positive frame vertices $\mathbf{p} \in \mathrm{fr}^+(\mathbf{f},\mathbf{s})$ only, since this trivially implies its own positivity.
- In the concluding remarks of [Fon+17] it is therefore left as a research question to find a linearization method that also allows the summation over variant negative frame vertices as long as the overall value of L_{k_1} is positive. In Subsection 2.3.2 we propose a novel linearization $\Phi^{\mathtt{Wk}}_{\mathtt{Constr}}$ that solves this issue with a more sophisticated analysis of the projected frame vertices. It increases the completeness of the original method while its consistency check is still feasible in a reasonable amount of time.

2.3.1. Strictly separable frame vertices

For a frame vertex $\mathbf{p} \in \mathrm{fr}(\mathbf{f})$ the sign of $\mathbf{s}^{\mathbf{p}}$ is fully determined by only those s_i with an odd exponent p_i. Treating negative signs as `True` and encoding the sign variant accordingly as a Boolean vector, it can be caclulated as the parity of the individual signs by the formula

$$\Phi_{\mathtt{Sgn}}(\mathbf{s} \mid \mathbf{p}) := \bigoplus_{\substack{i=1,\dots,d,\\ p_i \text{ odd}}} s_i.$$

Since the coefficients $f_{\mathbf{p}}$ are already known constants at the time of linearization, we can encode the membership $\mathbf{p} \in \mathrm{fr}^+(\mathbf{f}, \mathbf{s})$ as one static branch of the following case distinction

$$\Phi_{\mathtt{PosFrm}}(\mathbf{s} \mid \mathbf{f}, \mathbf{p}) := \begin{cases} \neg\Phi_{\mathtt{Sgn}}(\mathbf{s} \mid \mathbf{p}) & \text{, if } f_{\mathbf{p}} > 0, \\ \Phi_{\mathtt{Sgn}}(\mathbf{s} \mid \mathbf{p}) & \text{, if } f_{\mathbf{p}} < 0. \end{cases}$$

The coefficient $L_{k_1} = \sum_{\mathbf{n}\cdot\mathbf{p}=k_1} \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}}$ with the largest distance $k_1 := \max_{\mathbf{p}\in\mathrm{fr}(\mathbf{f})} \mathbf{n} \cdot \mathbf{p}$ solely consists of positive frame vertices if and only if there exists a threshold $b \in \mathbb{R}$ such that

(i) for all negative frame vertices $\mathbf{p} \in \mathrm{fr}^-(\mathbf{f}, \mathbf{s})$ it holds that $\mathbf{n} \cdot \mathbf{p} \leq b$, and

(ii) there exists at least one positive frame vertex $\mathbf{q} \in \mathrm{fr}^+(\mathbf{f}, \mathbf{s})$ with $\mathbf{n} \cdot \mathbf{q} > b$.

By strictly separating at least one positive from all negative frame vertices, these conditions test whether all frame vertices $\mathbf{p} \in \mathrm{fr}(\mathbf{f})$ with a distance $\mathbf{n} \cdot \mathbf{p} = k > b$ are positive. This implies that especially all the frame vertices with the largest existing distance $k_1 > b$ out of which L_{k_1} is composed of must be positive as well. The following formula directly encodes these conditions and can be handed over to a LRA solver for a consistency check:

$$\Phi^{\mathtt{Str}}_{\mathtt{Constr}}(\mathbf{s}, \mathbf{n}, b \mid \mathbf{f} > 0) := \left[\bigwedge_{\mathbf{p}\in\mathrm{fr}(\mathbf{f})} \Phi_{\mathtt{PosFrm}}(\mathbf{s} \mid \mathbf{f}, \mathbf{p}) \vee \mathbf{n} \cdot \mathbf{p} \leq b\right] \wedge \left[\bigvee_{\mathbf{q}\in\mathrm{fr}(\mathbf{f})} \Phi_{\mathtt{PosFrm}}(\mathbf{s} \mid \mathbf{f}, \mathbf{q}) \wedge \mathbf{n} \cdot \mathbf{q} > b\right].$$

Note that we face a linear real arithmetic problem, since we do not insist $\mathbf{n} = (n_1, \ldots, n_d)$ to be a vector of integral variables. A given solution for a linear formula stays valid even if all variable values are scaled by a common factor. If this linearization is satisfiable, we simply scale the resulting normal vector assignment to get back an integral solution.

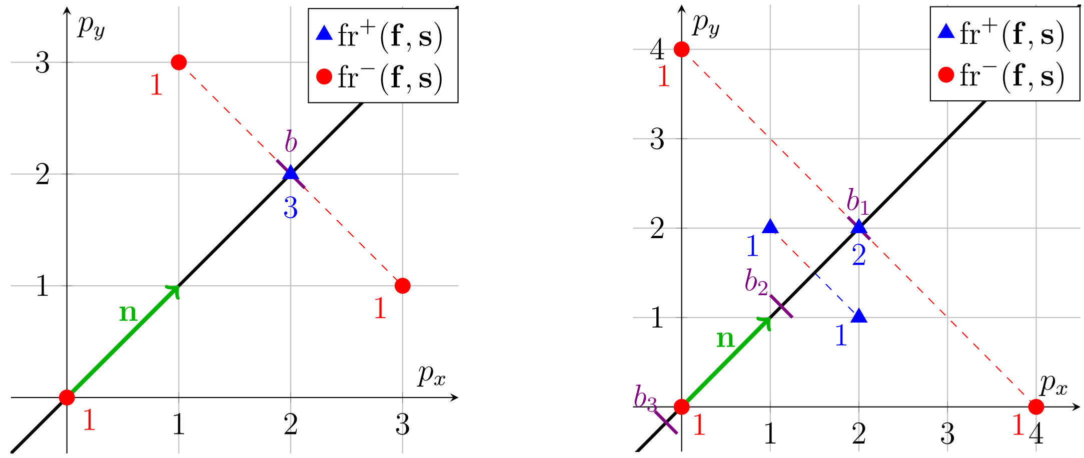
Example 2.5 For our running Example 2.4, we have marked a possible choice for a threshold b in Subfigure 2.1b. It separates the projected positive frame vertex $\mathbf{p} = (1, 3)$ from all negative frame vertex projections and hence proves that the coefficient L_{k_1} with $k_1 > b$ must be composed of positive frame vertices only.

2.3.2. Weakly separable frame vertices

As already mentioned, the coefficient $L_{k_1} = \sum_{\mathbf{n}\cdot\mathbf{p}=k_1} \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}}$ does not need to consist of variant positive frame vertices only to have a positive overall value:

Example 2.6 Consider the bivariate polynomial

$$\mathbf{f}(x, y) := -x^3y + 3x^2y^2 - xy^3 - 1 \in \mathbb{R}[x, y]$$



(a) Weakly separable frame vertices of the polynomial $f(x, y) := -x^3y + 3x^2y^2 - xy^3 - 1$.

(b) Linearly inseparable frame of the polynomial $f(x, y) := -x^4 + 2x^2y^2 - y^4 + xy^2 + x^2y - 1$.

Figure 2.2.: Projections onto the plane given by $\mathbf{n} = (1, 1)$ with sign variant $\mathbf{s} = (1, 1)$

whose frame is visualized in Subfigure 2.2a for a fixed sign variant $\mathbf{s} = (1, 1)$. There is no choice for a normal vector $\mathbf{n}$ such that L_{k_1} consists of variant positive frame vertices only. However, if we choose the normal vector $\mathbf{n} = (1, 1)$, then we get

$$L_{k_1} = f_{(3,1)} + f_{(2,2)} + f_{(1,3)} = -1 + 3 - 1 = 1 > 0$$

which is still positive although we have negative frame vertex contributions.

The idea for a more sophisticated linearization lies in a better analysis of the frame vertices projected onto the threshold border $b \in \mathbb{R}$ in the last subsection. We similarly claim that

(i') for all negative frame vertices $\mathbf{p} \in \text{fr}^-(\mathbf{f}, \mathbf{s})$ it holds that $\mathbf{n} \cdot \mathbf{p} \leq b$, and

(ii') there exists at least one positive frame vertex $\mathbf{q} \in \text{fr}^+(\mathbf{f}, \mathbf{s})$ with $\mathbf{n} \cdot \mathbf{q} \geq b$.

Notice the decisive difference in (ii') compared to the original condition (ii), where we now allow the projected frame vertex to lie on the threshold border through the use of a weak relation. The situation on the threshold border needs additional attention:

(a) If there exists a positive frame vertex $\mathbf{q} \in \text{fr}^+(\mathbf{f}, \mathbf{s})$ with $\mathbf{n} \cdot \mathbf{q} > b$, then the strict separability conditions (i) and (ii) of Subsection 2.3.1 are satisfied and L_{k_1} is positive.

(b) Otherwise we also have $\mathbf{n} \cdot \mathbf{q} \leq b$ for all variant positive frame vertices $\mathbf{q} \in \text{fr}^+(\mathbf{f}, \mathbf{s})$ and by condition (ii') there is at least one such vertex with $\mathbf{n} \cdot \mathbf{q} = b$. It follows that $k_1 = b$ and we therefore claim the coefficient $L_b = \sum_{\mathbf{n} \cdot \mathbf{p} = b} f_{\mathbf{p}}$ to be positive.

Both of these cases can be handled simultaneously by the following reformulated condition that furthermore allows a very elegant encoding into a linear real arithmetic formula:

(iii') The *total rating* of $\mathbf{f}(\mathbf{x})$ at the threshold border b is defined by

$$\mathbf{r}(\mathbf{f} \mid \mathbf{s}, \mathbf{n}, b) := \sum_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} r_{\mathbf{p}} \qquad \text{with} \qquad r_{\mathbf{p}} := \begin{cases} +\infty & \text{, if } \mathbf{n} \cdot \mathbf{p} > b, \\ \mathbf{s}^{\mathbf{p}} f_{\mathbf{p}} & \text{, if } \mathbf{n} \cdot \mathbf{p} = b, \\ 0 & \text{, if } \mathbf{n} \cdot \mathbf{p} < b. \end{cases}$$

Suppose that the weak separability conditions (i') and (ii') above are given. Then the two cases (a) and (b) are equivalent to the inequality $\mathbf{r}(\mathbf{f} \mid \mathbf{s}, \mathbf{n}, b) > 0$.

Treating the $r_{\mathbf{p}}$ as additional indeterminates, an encoding of condition (iii') is given by

$$\Phi_{\mathtt{Rtg}}(\mathbf{s}, \mathbf{n}, b, \mathbf{r} \mid \mathbf{f}) := \sum_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} r_{\mathbf{p}} > 0 \wedge \left[\bigwedge_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} \mathbf{n} \cdot \mathbf{p} < b \to r_{\mathbf{p}} = 0 \right] \wedge$$
$$\left[\bigwedge_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} \mathbf{n} \cdot \mathbf{p} = b \to (\Phi_{\mathtt{Sgn}}(\mathbf{s} \mid \mathbf{p}) \wedge r_{\mathbf{p}} = -f_{\mathbf{p}}) \vee (\neg \Phi_{\mathtt{Sgn}}(\mathbf{s} \mid \mathbf{p}) \wedge r_{\mathbf{p}} = f_{\mathbf{p}}) \right],$$

Note that the remaining case $\mathbf{n} \cdot \mathbf{p} > b \to r_{\mathbf{p}} = +\infty$ is superfluous and deliberately not encoded to reduce the size of the linearization. If the premise $\mathbf{n} \cdot \mathbf{p} > b$ is satisfied, then the variable $r_{\mathbf{p}}$ is not fixed by the above formula. The LRA solver is able to assign an arbitrarily large value to $r_{\mathbf{p}}$ in order to fulfill the total rating constraint $\sum_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} r_{\mathbf{p}} > 0$, which would also be the desired effect of the conclusion $r_{\mathbf{p}} = +\infty$. The full linearization taking also the weak separability conditions (i') and (ii') into account is now given by

$$\Phi^{\mathtt{Wk}}_{\mathtt{Constr}}(\mathbf{s}, \mathbf{n}, b, \mathbf{r} \mid \mathbf{f} > 0) := \Phi_{\mathtt{Rtg}}(\mathbf{s}, \mathbf{n}, b, \mathbf{r} \mid \mathbf{f}) \wedge$$
$$\left[\bigwedge_{\mathbf{p} \in \mathrm{fr}(\mathbf{f})} \Phi_{\mathtt{PosFrm}}(\mathbf{s} \mid \mathbf{f}, \mathbf{p}) \vee \mathbf{n} \cdot \mathbf{p} \leq b \right] \wedge \left[\bigvee_{\mathbf{q} \in \mathrm{fr}(\mathbf{f})} \Phi_{\mathtt{PosFrm}}(\mathbf{s} \mid \mathbf{f}, \mathbf{q}) \wedge \mathbf{n} \cdot \mathbf{q} \geq b \right].$$

A brief inspection of this formula shows that condition (ii') is already included in condition (iii'): If there is no positive frame vertex $\mathbf{q} \in \mathrm{fr}^{+}(\mathbf{f}, \mathbf{s})$ with $\mathbf{n} \cdot \mathbf{q} \geq b$, then the total rating of $\mathbf{f}(\mathbf{x})$ cannot be positive. Eliminating this redundancy from the linearization however lead to an extraordinary increase of the runtime for its consistency check in all our experiments. In case of a conflict the total rating formula $\Phi_{\mathtt{Rtg}}(\mathbf{s}, \mathbf{n}, b, \mathbf{r} \mid \mathbf{f})$ provides very few information for its resolution. The redundant encoding of condition (ii') excludes obviously unsatisfiable choices for the indeterminates $\mathbf{s}$, $\mathbf{n}$ and b before the total rating formula gets evaluated.

Example 2.7 Reconsider the Example 2.6 and its visualization in Subfigure 2.2a. There is no threshold b that strictly separates at least one positive frame vertex from all negative frame vertices. But if we choose the threshold $b = 4$, then we get the vertex ratings

$r_{(1,3)} = -1$, $r_{(2,2)} = 3$, $r_{(3,1)} = -1$, and $r_{(0,0)} = 0$. The total rating of $\mathbf{f}(\mathbf{x})$ is therefore given by $\mathbf{r}(\mathbf{f} \mid \mathbf{s}, \mathbf{n}, b) = L_{k_1} = 1$ which fulfills the total rating condition (iii').

2.3.3. Linearly inseparable frame vertices

As already mentioned, the coefficient L_{k_1} corresponding to the largest projection length k_1 does not necessarily represent the leading coefficient of $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n})$, since it may vanish.

Example 2.8 Consider the visualization of the bivariate polynomial

$$f(x, y) := -x^4 + 2x^2y^2 - y^4 + xy^2 + x^2y - 1 \in \mathbb{R}[x, y]$$

in Subfigure 2.2b. If we choose $\mathbf{n} = (1, 1)$ and $\mathbf{s} = (1, 1)$, then we get

- $L_4 = f_{(4,0)} + f_{(2,2)} + f_{(0,4)} = -1 + 2 - 1 = 0$,
- $L_3 = f_{(1,2)} + f_{(2,1)} = 1 + 1 = 1$,
- $L_0 = f_{(0,0)} = -1$,

and the full Laurent polynomial is given by $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n}) = t^3 - 1$. The true leading coefficient L_3 is positive, but the coefficient L_4 with the largest projective distance vanishes. Choosing another sign variant $\mathbf{s}$ is not an option, since the variant negative frame $\mathrm{fr}^-(\mathbf{f}, \mathbf{s})$ consists of vertices with even parity and hence is invariant to sign changes.

This problem can get even worse if not only L_{k_1} is zero, but a whole sequence of coefficients L_{k_i} for $i = 1, 2, \ldots$ up to the point where the whole Laurent polynomial vanishes. To overcome this issue, we were able to find a linearization that starting with $i = 1, 2, \ldots$

(i) tests whether $L_{k_i} > 0$ with a threshold border b_i using the weak separability method,

(ii) in case $L_{k_i} = 0$ uses the threshold b_i to discard all frame vertices $\mathbf{p} \in \mathrm{fr}(\mathbf{f})$ with $\mathbf{n} \cdot \mathbf{p} \geq b$ for the next iteration of the weak separability method.

The consistency check of the resulting linearization was infeasible in a reasonable amount of time even on hand-crafted toy examples with only three variables. This generalization was therefore discarded from our final `STropModule` code base.

2.4. Application to the SMT problem

The so far presented approaches identify real-valued assignments, where only a single multivariate polynomial becomes positive. We will use the derived linearizations in the strict and weak encoding variants $\Phi^{\mathtt{Str}}_{\mathtt{Constr}}(\mathbf{s}, \mathbf{n}, b \mid \mathbf{f} > 0)$ and $\Phi^{\mathtt{Wk}}_{\mathtt{Constr}}(\mathbf{s}, \mathbf{n}, b, \mathbf{r} \mid \mathbf{f} > 0)$, respectively, as interchangeable building blocks to solve the DPLL based SMT Problem. Let us denote any of these two formulas by $\Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid \mathbf{f} > 0)$ for convenience.

2.4.1. Single constraint with an arbitrary relation

A single constraint $\mathbf{f}(\mathbf{x}) \sim 0$ with an arbitrary relation symbol $\sim \in \{<, \leq, =, \neq, \geq, >\}$ can be reduced to the already known case by applying the following rewriting rules:

$$\Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid \mathbf{f} \sim 0) := \begin{cases} \Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid -\mathbf{f} > 0) & \text{, if } \sim \in \{<, \leq\} \\ \Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid -\mathbf{f} > 0) \\ \qquad \vee \Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid \mathbf{f} > 0) & \text{, if } \sim \in \{\neq\} \\ \Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid \mathbf{f} > 0) & \text{, if } \sim \in \{\geq, >\} \end{cases}$$

It is worth mentioning that the relation symbols $\{<, \leq\}$ and $\{\geq, >\}$ define classes with the same limiting behaviour and hence are mapped to the same linearization. Furthermore, the lack of a rewriting rule for the case of an equality relation $=$ is no mistake: As seen in Section 2.3 our linearization method is based on the linear separability of positive and negative frame vertices. But rewriting $\mathbf{f}(\mathbf{x}) = 0$ as the conjunction $-\mathbf{f}(\mathbf{x}) \geq 0 \wedge \mathbf{f}(\mathbf{x}) \geq 0$ and encoding both clauses independently from each other will lead to a linearization

$$\Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid -\mathbf{f} > 0) \wedge \Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid \mathbf{f} > 0)$$

that is always inconsistent: By the coincidence $\mathrm{fr}^+(\mathbf{f}, \mathbf{s}) = \mathrm{fr}^-(-\mathbf{f}, \mathbf{s})$ there is no normal vector $\mathbf{n}$ such that the linear separability conditions in Subsections 2.3.1 and 2.3.2 are fulfilled. But from the unsatisfiability of this linearization we are unable to draw any conclusions about the consistency of the original constraint $\mathbf{f}(\mathbf{x}) = 0$ without the aid of backend solvers, since the limiting behaviour analysis is insufficient to exclude solutions within a bounded support. Hence, the presence of equality constraints leads to an immediate abort of our subtropical method with an `Unknown` result. In every other case, provided the consistency check verifies the satisfiability of the linearization $\Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b[, \mathbf{r}] \mid \mathbf{f} \sim 0)$, a solution for the original constraint $\mathbf{f}(\mathbf{x}) \sim 0$ can be reconstructed as the image of the momentum curve $\mathbf{m}(t \mid \mathbf{s}, \mathbf{n})$ for a large enaugh $t \in \mathbb{R}_+$, where $\mathbf{L}(t \mid \mathbf{f}, \mathbf{s}, \mathbf{n}) \sim 0$ is fulfilled.

2.4.2. Common solution of multiple constraints

Let a constraint conjunction $\mathbf{C} = \bigwedge_{i=1}^{m} \mathbf{f}_i(\mathbf{x}) \sim_i 0$ with polynomials $\mathbf{f}_i(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_d]$ and relations $\sim_i \in \{<, \leq, =, \neq, \geq, >\}$ be given. If the independently linearized formulas $\Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b_i[, \mathbf{r}_i] \mid \mathbf{f}_i \sim_i 0)$ share their sign variant $\mathbf{s} \in \{\pm 1\}^d$ and their normal vector

$\mathbf{n} \in \mathbb{R}^d$, then in case of satisfiability a common solution for all constraints will be given by $\mathbf{m}(t \mid \mathbf{s}, \mathbf{n})$ for a large enaugh $t \in \mathbb{R}_+$, where $\mathbf{L}(t \mid \mathbf{f}_i, \mathbf{s}, \mathbf{n}) \sim_i 0$ for all $i = 1, \ldots, m$ is fulfilled. We hence linearize $\mathbf{C}$ by

$$\Phi_{\texttt{SMT}}(\mathbf{s}, \mathbf{n}, b_1, \ldots, b_m[, \mathbf{r}_1, \ldots, \mathbf{r}_m] \mid \bigwedge_{i=1}^{m} \mathbf{f}_i \sim_i 0) := \bigwedge_{i=1}^{m} \Phi_{\texttt{Constr}}(\mathbf{s}, \mathbf{n}, b_i[, \mathbf{r}_i] \mid \mathbf{f}_i \sim_i 0).$$

In addition to the already described short-circuiting for equality constraints in Subsection 2.4.2, the combination of multiple constraints can help avoiding a consistency check in many more cases. For this purpose we normalize the left hand side of every constraint $\mathbf{f}_i(\mathbf{x}) \sim_i 0$ by enforcing a unit leading coefficient and turning the relation symbol accordingly. Constraints with the same left hand side $\mathbf{f}(\mathbf{x})$ but different relations are

unsatisfiable if the relations contradict each other in the cases $\{<, =\}$, $\{<, >\}$, $\{<, \geq\}$, $\{\leq, >\}$ and $\{=, >\}$. From contradicting relations $\{\sim_1, \sim_2\}$ an unsatisfiable core $\mathbf{f}(\mathbf{x}) \sim_1 0 \wedge \mathbf{f}(\mathbf{x}) \sim_2 0$ can be generated before terminating with an `Unsat` result.

undecidable if the relations resemble an equality in the cases $\{=\}$ and $\{\leq, \geq\}$. We therefore skip the application of the subtropical method and directly call the backend solvers of the defined strategy tree on the given constraint conjunction $\mathbf{C}$.

Interestingly, when these two sources for a fast skip of the subtropical method are excluded, only a single relation for the same left hand side $\mathbf{f}(\mathbf{x})$ can be active. Take as an example the relations $\{\neq, \geq, >\}$, where we only need to use the strictest relation $>$ for a linearization. These simple optimizations accelerated the consistency check on the benchmarks presented in Section 3.5 by more than ten times. We discovered that a large number of the constraint conjunctions given to our `STropModule` can be decided or skipped by these simple pre-tests without an invocation of the LRA solver on the linearizations.

2.4.3. Extension to mixed-integer problems

The so far presented subtropical method is defined as a lightweight decision procedure for real arithmetic problems only. We propose the following extension to the original algorithm that works pretty well on mixed-integer problems with a small number of integer-valued coordinates. Remember that in case of satisfiability of the constructed linearization $\Phi_{\texttt{SMT}}(\mathbf{s}, \mathbf{n}, \{b_i[, \mathbf{r}_i]\}_{i=1}^{m} \mid \{\mathbf{f}_i \sim_i 0\}_{i=1}^{m})$ a corresponding variable assignment for the original constraint conjunction $\mathbf{C} = \bigwedge_{i=1}^{m} \mathbf{f}_i(\mathbf{x}) \sim_i 0$ is given by the image of the momentum curve

$$\mathbf{a} := \mathbf{m}(t \mid \mathbf{s}, \mathbf{n}) = (s_1 t^{n_1}, \ldots, s_d t^{n_d}) \qquad \text{for} \qquad t \gg 1.$$

A coordinate x_i will receive an integral solution a_i if and only if we can ensure the integrity of the expression $s_i t^{n_i}$. As a sufficient condition, we enforce the normal vector component n_i and the sample point t to take non-negative integral values. We therefore assert

$$\Phi_{\texttt{Int}}(\mathbf{n} \mid \mathbf{x}) := \bigwedge_{\substack{i=1,\dots,d,\\ x_i \text{ integral}}} n_i \geq 0$$

and only use test points $t \in \mathbb{N}$ for the later variable assignment reconstruction. Restricting a single normal vector value n_i to the positive axis effectively halves the solution search space for the linearization. We hence note that with a growing number of integer-valued coordinates x_i our `STropModule` is less likely to give a satisfiable answer.

2.5. Benchmarking results and conclusion

We tested our `STropModule` implementation on the QF_NRA division of the `SMT-LIB` benchmarking library [BFT16] stemming from the industrial and academic world. Every benchmark consists of an input formula and the expected answer of its consistency check, if the status is already defined. They are grouped into families of similar problems like termination proofs or elementary function approximations and have a variable complexity regarding the formula sizes and the number of variables. We performed the experiments in Table 2.1 on a 2.7 GHz Intel Core i7-4800MQ CPU with a timeout of 60 seconds and 4 GB memory per benchmark. If any of these resource limits is exceeded, the corresponding run gets terminated with a `Resout` answer by our benchmarking scheduler. Otherwise, we report the number of benchmarks (Num) and the avarage runtime (Avg) in milliseconds for each possible answer `Sat`, `Unsat` or `Unknown` and for each of the following strategies:

STrop only: `SATModule`→`STropModule`

Backends only: `SATModule`→`ICPModule`→`VSModule`→`CADModule`

STrop + Backends: `SATModule`→`STropModule`→`ICPModule`→`VSModule`→`CADModule`

The strict and weak variants denote the two linearization methods from Section 2.3.

The strategy that combines the strict variant of our `STropModule` with the standard backends consistently outperforms the pure backends on almost every benchmark family. Even for those benchmarks, where the number of answers of one class are kept stable, we observe a considerable decrease of the average runtime. As an example for this behaviour, compare the `Sat` answers for the kissing family or the `Unsat` answers for the Sturm MBO/MGC family. The later shows another interesting property of DPLL based theory solvers in general: Although the subtropical method only focuses on proving satisfiability, it is able to accelerate also unsatisfiable answer deductions. It happens that many of the partial CNF formulas passed to the theory solver are satisfiable, even if the complete formula that gets checked is unsatisfiable. On the remaining instances, our `STropModule` fails quickly with an `Unknown` result and thus generates a very small overhead. To see this, consider the pure subtropical method on the Heizmann Ultimate Invariant Synthesis and Ultimate Automizer families. It is able to generate `Unknown` answers, where the full strategies timeout, proving its efficiency. For the final question which linearization variant to use, consider the meti-tarski and zankl families: As a theory solver on its own, the additional completeness of the weak variant comes with a 25 fold and a 84 fold increase of the average runtime, respectively. If it is used upfront to more sophisticated theory solvers for the nonlinear real arithmetic, its linearization complexity degrades the performance of the full strategy compared to the strict variant. In an environment like `SMT-RAT`, where already more complete decision procedures exist, we therefore highly recommend the strict variant as a lightweight heuristic to decide simpler input instances.

QF_NRA Benchmarks		**STrop strict only**		**STrop weak only**		**STrop strict + Backends**		**STrop weak + Backends**		**Backends only**	
Benchmark Family	Answer	Num	Avg	Num	Avg	Num	Avg	Num	Avg	Num	Avg
Sturm MBO/MGC (414)	`Sat` (107)	0	0	0	0	0	0	0	0	0	0
	`Unsat` (292)	2	9	2	9	122	5556	122	5748	122	6035
	`Unknown` (15)	412	7475	412	7722	0	0	0	0	0	0
	`Resout`	0	0	0	0	292	60049	292	60078	292	60042
Heizmann Ultimate Invariant Synthesis (69)	`Sat` (0)	0	0	0	0	0	0	0	0	0	0
	`Unsat` (0)	0	0	0	0	0	0	0	0	0	0
	`Unknown` (69)	51	4255	51	4466	0	0	0	0	0	0
	`Resout`	18	60156	18	60032	69	59987	69	59994	69	59975
hong (20)	`Sat` (0)	0	0	0	0	0	0	0	0	0	0
	`Unsat` (20)	0	0	0	0	20	221	20	273	20	15
	`Unknown` (0)	20	206	20	239	0	0	0	0	0	0
	`Resout`	0	0	0	0	0	0	0	0	0	0
hycomp (2752)	`Sat` (191)	0	0	0	0	24	14801	23	13280	26	14565
	`Unsat` (2191)	1898	1419	1898	1507	1804	3196	1798	3116	1783	3033
	`Unknown` (370)	19	5220	19	5505	0	0	0	0	0	0
	`Resout`	835	57952	835	58011	924	57992	931	58069	943	58038
kissing (45)	`Sat` (42)	0	0	0	0	10	127	10	134	10	147
	`Unsat` (3)	0	0	0	0	0	0	0	0	0	0
	`Unknown` (0)	45	41	45	41	0	0	0	0	0	0
	`Resout`	0	0	0	0	35	59989	35	59991	35	59975

LassoRanker (821)	`Sat` (121)	0	0	0	0	0	0	0	0	0	0
	`Unsat` (133)	0	0	0	0	0	0	0	0	0	0
	`Unknown` (567)	0	0	0	0	0	0	0	0	0	0
	`Resout`	821	60020	821	60012	821	59995	821	60005	821	60005
meti-tarski (7006)	`Sat` (4391)	1277	11	1346	281	4179	184	4176	366	4169	228
	`Unsat` (2615)	703	9	703	8	2340	274	2343	283	2339	257
	`Unknown` (0)	5026	10	4955	123	0	0	0	0	0	0
	`Resout`	0	0	2	60006	487	58552	487	58600	498	58624
Ultimate Automizer (61)	`Sat` (48)	0	0	0	0	0	0	0	0	0	0
	`Unsat` (13)	0	0	0	0	0	0	0	0	0	0
	`Unknown` (0)	44	2989	44	3058	0	0	0	0	0	0
	`Resout`	17	60036	17	60027	61	59992	61	59998	61	59982
zankl (166)	`Sat` (63)	31	84	28	7082	46	1650	40	4057	22	4955
	`Unsat` (29)	2	15	2	8	16	321	16	666	16	605
	`Unknown` (74)	94	2024	76	1913	0	0	0	0	0	0
	`Resout`	39	60023	60	60018	104	59458	110	59185	128	59493
Total (11354)	`Sat` (4963)	1308	12	1374	419	4259	283	4249	470	4227	340
	`Unsat` (5296)	2605	1036	2605	1100	4302	1649	4299	1624	4280	1578
	`Unknown` (1095)	5711	661	5622	784	0	0	0	0	0	0
	`Resout`	1730	59024	1753	59060	2793	59066	2806	59094	2843	59174

Table 2.1.: Benchmarking results of the `STropModule` on the QF_NRA division of the `SMT-LIB`.

Chapter 3.

The Case-Splitting Method

In this chapter we illustrate the implemented functionality behind our case-splitting module (`CSplitModule`) that is presented in Appendix B. In their first publication [Bor+09] Borralleras et al. proposed a reduction method for nonlinear to linear integer arithmetic formulas generalizing the idea previously known from bit-blasting to a higher-order target logic. It is based on the linearization of nonlinear monomials by a repeated application of a case analysis on the possible values that some of the variables in the monomial can take. To ensure completeness, this method requires the domains of variables used for case-splits to be finite. In reality, this basic idea quickly loses its termination power for bounded, but large variable domains. In the follow-up paper [Bor+12] this issue is addressed by replacing the unary encoding of large domains through an improved encoding in a positional numeral system. Additionally, the authors present a novel method to handle entirely unbounded domains via an incremental approach to introduce and expand artificial bounds in a clever way. This allows us to give unsatisfiability answers for certain input formulas even in the later case and therefore improves the completeness of the original method.

3.1. Case-splits for monomial equalities

The main idea of the case-splitting method is the linearization of nonlinear monomials by a repeated application of a case analysis on the possible values that some of the variables in the monomial can take. Consider for example the nonlinear integer arithmetic formula

$$x = abc \quad \wedge \quad 5 \leq x \leq 10 \quad \wedge \quad 2 \leq a, b, c \leq 3.$$

Since the variable a is integral, it can only take the values $a = 2$ or $a = 3$, and we get an equisatisfiable formula by replacing the constraint $x = abc$ by a simple case distinction as

$$\begin{aligned} y = bc \quad \wedge \quad a = 2 \rightarrow x = 2y \quad \wedge \quad 5 \leq x \leq 10 \\ \wedge \quad a = 3 \rightarrow x = 3y \quad \wedge \quad 2 \leq a, b, c \leq 3. \end{aligned}$$

Note the additional substitution of the nonlinear expression bc by a fresh intermediate variable y. This produces a new nonlinear equality $y = bc$ with a smaller total degree, and appart from that linear constraints only. Another case-split on the variable b gives

$$\begin{aligned} & b = 2 \to y = 2c \quad \wedge \quad a = 2 \to x = 2y \quad \wedge \quad 5 \le x \le 10 \\ \wedge \quad & b = 3 \to y = 3c \quad \wedge \quad a = 3 \to x = 3y \quad \wedge \quad 2 \le a, b, c \le 3, \end{aligned}$$

which can be checked by a linear integer arithmetic solver such as the Branch-and-Bound method for satisfiability. The resulting model $\{a = 2, b = 2, c = 2, y = 4, x = 8\}$ also satisfies the original formula, where the additional assignment $y = 4$ for the intermediate variable can be dropped. This trivial example points to the main challenges of the method:

- In the foregoing example, there were only two possible values for the variables a and b used for case distinctions. In the following sections, we will develop improved linearization techniques for case variables with large but bounded domains.
- If the domains of variables used for case distinctions are even unbounded, we need to introduce new bounds and hence we lose completeness. A proper analysis of the unsatisfiable core of the background linear arithmetic solver however will allow us to prove unsatisfiability even in this case. Furthermore, it will yield an efficient incremental method to choose the variable bounds that need to be enlarged in order to continue the search for a satisfying assignment for the original set of constraints.

In Section 3.2 we first describe some lightweight preprocessing steps to extract monomial equalities from a given conjunction of constraints. The remaining sections are then devoted to the linearization of those nonlinear monomial equalities.

3.2. Purification of nonlinear constraints

Let $\mathbf{C} := \bigwedge_{i=0}^{m} \mathbf{f}_i(\mathbf{x}) \sim_i 0$ be a conjunction of constraints with multivariate polynomials $\mathbf{f}_i(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_d]$ and relations $\sim_i \in \{<, \le, =, \neq \ge, >\}$. Consider the mixed-integer problem of finding a satisfying assignment $\mathbf{a} \in \mathbb{R}^k \times \mathbb{Z}^{d-k}$.

3.2.1. Discretization of real-valued variables

Since the main case-splitting method is designed to solve integer arithmetic problems only, we first need to get rid of the real-valued variables $x_1, \ldots, x_k$. In [Bor+12] several discretization techniques are considered:

Constant Denominator Fix a common denominator $D \in \mathbb{Z}$, $D \neq 0$. For $i = 1, \ldots, k$

choose fresh integer-valued variables n_i and perform the substitution $x_i := \frac{n_i}{D}$. Whenever this method is applied, we definitely lose completeness by excluding point solutions that cannot be written as a quotient with denominator D.

Full Quotient For $i = 1, \dots, k$ choose fresh integer-valued variables n_i and d_i. Perform the substitution $x_i := \frac{n_i}{d_i}$ and eliminate the denominators in $\mathbf{C}$ by multiplying the constraints with a power of d_i to get back an integer arithmetic formula.

Although the full quotient method is more expressive, the explosion of the monomial degrees turned out to be computationally infeasible in all our experiments. The number of intermediate variables needed to linearize the formula results in a very poor performance of the linear arithmetic solver. This confirms the concerns in [Bor+12] regarding this approach. For our `CSplitModule` the constant denominator approach was therefore finally chosen. From now on, let $\mathbf{C}$ denote an already discretized integer arithmetic problem.

3.2.2. Extraction of nonlinear monomial equations

For each constraint $\mathbf{f}(\mathbf{x}) \sim 0$ in $\mathbf{C}$ we perform the following preprocessing steps: Replace every nonlinear monomial $\mathbf{x}^{\mathbf{p}}$ in $\mathbf{f}(\mathbf{x})$ with $d_{\mathbf{p}} := \|\mathbf{p}\|_1 > 1$ by a fresh integer-valued variable $y_{\mathbf{p}}$ to get the *linear part* of the original constraint by

$$\mathbf{L}(\mathbf{f}(\mathbf{x}) \sim 0) := \sum_{\substack{\mathbf{p} \in \mathrm{fr}(\mathbf{f}) \\ d_{\mathbf{p}} \leq 1}} f_{\mathbf{p}} \mathbf{x}^{\mathbf{p}} + \sum_{\substack{\mathbf{p} \in \mathrm{fr}(\mathbf{f}) \\ d_{\mathbf{p}} > 1}} f_{\mathbf{p}} y_{\mathbf{p}} \sim 0.$$

We want the remaining *monomial equations* of the form $y_{\mathbf{p}} = \mathbf{x}^{\mathbf{p}}$ to have a total degree of two. For this purpose, choose an index $b_{\mathbf{p}} \in \{1, \dots, d\}$ with $p_{b_{\mathbf{p}}} > 0$ and perform the split into a binary equation $y_{\mathbf{p}} = x_{b_{\mathbf{p}}} \cdot z_{\mathbf{p}}$ and $z_{\mathbf{p}} = \mathbf{x}^{\mathbf{p} - \mathbf{e}_{b_{\mathbf{p}}}}$ for a fresh *intermediate variable* $z_{\mathbf{p}}$. Repeat this binarization process with $z_{\mathbf{p}} = \mathbf{x}^{\mathbf{p} - \mathbf{e}_{b_{\mathbf{p}}}}$ until all resulting monomial equations have a total degree of two. The depicted iteration constructs a so-called *reduction sequence* $\mathbf{b}_{\mathbf{p}} = (b_{\mathbf{p},1}, \dots, b_{\mathbf{p},d_{\mathbf{p}}})$ of indices $b_{\mathbf{p},j} \in \{1, \dots, d\}$ that decomposes the exponent into $\mathbf{p} = \sum_{j=1}^{d_{\mathbf{p}}} \mathbf{e}_{b_{\mathbf{p},j}}$. The *nonlinear part* of the constraint $\mathbf{f}(\mathbf{x}) \sim 0$ is then given by

$$\mathbf{N}(\mathbf{f}(\mathbf{x}) \sim 0) := \bigwedge_{\substack{\mathbf{p} \in \mathrm{fr}(\mathbf{f}) \\ d_{\mathbf{p}} > 1}} \left[\bigwedge_{j=1}^{d_{\mathbf{p}}-2} y_{\mathbf{p},j} = x_{b_{\mathbf{p},j}} \cdot y_{\mathbf{p},j+1} \right] \wedge y_{\mathbf{p},d_{\mathbf{p}}-1} = x_{b_{\mathbf{p},d_{\mathbf{p}}-1}} \cdot x_{b_{\mathbf{p},d_{\mathbf{p}}}}$$

for a sequences of intermediate variables $\mathbf{y}_{\mathbf{p}} = (y_{\mathbf{p},1}, \dots, y_{\mathbf{p},d_{\mathbf{p}}-1})$. The complexity of the final linearization is highly dependent on the right choice of these reduction sequences and we therefore devote Subsection 3.4.2 to this problem. The *purification* of the constraints conjunction **C** is now given by the decomposition

$$\mathbf{L} := \bigwedge_{i=1}^{m} \mathbf{L}(\mathbf{f}_i(\mathbf{x}) \sim_i 0) \qquad \text{and} \qquad \mathbf{N} := \bigwedge_{i=1}^{m} \mathbf{N}(\mathbf{f}_i(\mathbf{x}) \sim_i 0)$$

Note that $\mathbf{C}' := \mathbf{L} \wedge \mathbf{N}$ is still a conjunction of constraints and equisatisfiable to **C**. From now on, let $\mathbf{C} = \mathbf{C}'$ denote an already purified input formula in which the only nonlinearities arise as binary monomial equations of the form $x = v \cdot w$ in **N**.

3.3. Case-splitting for variables with bounded domains

Suppose that for every variable v in **C** we have a *maximal domain* $D_v := [\mathcal{L}_v, \mathcal{U}_v] \subseteq \mathbb{Z}$ that restricts its solution search space. In our `CSplitModule`, these domains are extracted directly from the input **C** exploiting linear *bounding constraints* of the form $v \sim 0$ in **L** with $\sim \in \{<, \leq, =, \geq, >\}$. For future development an enhanced bounds extraction routine should be implemented that is able to take also nonlinear constraints like $v^2 \leq 10$ into account. We fix a constant number $T \in \mathbb{N}$, $T \geq 2$, and subdivide all variable domains into the following classes: We denote D_v as *small* if $|D_v| \leq T$, *large* if $T < |D_v| < +\infty$, and *unbounded* otherwise. In this section, we present linearization techniques for the binary monomial equations $x = v \cdot w$ in **N** with bounded domains D_v.

3.3.1. Handling small domains

Suppose for the moment that the variable v used for case-splits has a small domain D_v. The simple linearization rule seen in the introductory Section 3.1 can be restated as

$$\Phi^{\texttt{Small}}_{\texttt{Monomial}}(x, v, w) := \bigwedge_{\alpha=\mathcal{L}_v}^{\mathcal{U}_v} (v = \alpha \rightarrow x = \alpha \cdot w)$$

For a single monomial equation $x = v \cdot w$ it produces $|D_v| \in \mathcal{O}(T)$ binary linear clauses and is therefore inappropriate for the linearization of large domains for $T \gg 1$. If the variable v is restricted to the domain D_v, then the monomial equation is equisatisfiable to its linearization. Remember, that this premise is fulfilled within the formula **C**, since the variable domain D_v was extracted from the bounding constraints contained in it.

3.3.2. Handling large domains

The problem with the presented linearization rule for small domains lies in the unary encoding of the variable domain $D_v = [\mathcal{L}_v, \mathcal{U}_v]$, where every binary clause represents a single value that the case variable v can take. To overcome this issue, we fix another integer $B \in \mathbb{N}$, $2 \leq B \leq T$, and subsequently encode the variable domain D_v in a positional numeral system to the base B. To this end, we take fresh integer-valued variables q and r and perform a symbolic division of v modulo B by introducing the linear formula

$$\Phi_{\texttt{Digit}}(v,q,r) := v = B \cdot q + r \wedge \lfloor \tfrac{\mathcal{L}_v}{B} \rfloor \leq q \leq \lfloor \tfrac{\mathcal{U}_v}{B} \rfloor \wedge 0 \leq r \leq B-1.$$

The added variable domains $D_q = [\lfloor \frac{\mathcal{L}_v}{B} \rfloor, \lfloor \frac{\mathcal{U}_v}{B} \rfloor]$ and $D_r = [0, B-1]$ constitute a smallest possible over-approximation of D_v in the usual sense of interval analysis: Every value $\hat{v} \in D_v$ can be written as a linear combination $\hat{v} = B \cdot \hat{q} + \hat{r}$ for suitable choices of $\hat{q} \in D_q$ and $\hat{r} \in D_r$, and the two domains D_q and D_r are minimal with respect to this condition. Hence, substituting the expression $B \cdot q + r$ for v in the original monomial equation $x = v \cdot w$ does not exclude any solutions and we obtain

$$x = v \cdot w = (B \cdot q + r) \cdot w = B \cdot q \cdot w + r \cdot w.$$

In order to get rid of the remaining nonlinear monomials on the right hand side, we

- replace $q \cdot w$ by a fresh integer-valued variable y and get a new monomial equation $y = q \cdot w$ that still needs to be linearized. But the qualitative difference between the later and the original constraint $x = v \cdot w$ is the domain size reduction $|D_q| \leq \lceil \frac{|D_v|}{B} \rceil$.
- perform an unary case-split on the monomial $r \cdot w$ using r as the case variable like in Subsection 3.3.1. Note that the variable domain D_r is small since $|D_r| = B - 1 \leq T$.

In summary, we replace the monomial equation $x = v \cdot w$ with the linearization

$$\Phi_{\texttt{Expansion}}(x,y,v,q,r,w) := \bigwedge_{\alpha=0}^{B-1} (r = \alpha \rightarrow x = B \cdot y + \alpha \cdot w) \wedge \Phi_{\texttt{Digit}}(v,q,r)$$

and repeat this linearization process on the remaining constraint $y = q \cdot w$. In each iteration, the domain size of D_q is reduced by a factor of B. After at most $k := \lceil \log_B |D_v| \rceil$ iterations, the variable domain D_q is small and we encode the constraint $y = q \cdot w$ as seen in Subsection 3.3.1. To formalize this described iteration process, choose sequences of integer-valued variables $\mathbf{x} = (x_0, \ldots, x_k)$, $\mathbf{q} = (q_0, \ldots, q_k)$ and $\mathbf{r} = (r_1, \ldots, r_k)$. With the identification $x_0 := x$ and $q_0 := v$, the full linearization of $x = v \cdot w$ is given by

$$\Phi^{\texttt{Large}}_{\texttt{Monomial}}(\mathbf{x},\mathbf{q},\mathbf{r},w) := \bigwedge_{i=1}^{k} \Phi_{\texttt{Expansion}}(x_{i-1},x_i,q_{i-1},q_i,r_i,w) \wedge \Phi^{\texttt{Small}}_{\texttt{Monomial}}(x_k,q_k,w)$$

For a single monomial equation $x = v \cdot w$ with bounded domain D_v this linearization rule produces $\mathcal{O}(B \log_B |D_v| + T)$ at most binary linear clauses. Suppose that the pure nonlinear formula $\mathbf{C}$ is transformed into $\mathbf{C}'$ by one application of this linearization rule. Since the variable domain D_v was derived from $\mathbf{C}$, both formulas are equisatisfiable.

3.4. Unsatisfiability and learning for unbounded domains

As one source of incompleteness of the case-splitting method we have already identified the discretization of real-valued variables. If a case variable used for linearization lacks a finite upper or lower bound, then we have to introduce artificial bounds and again we lose completeness at first glance. In this section, we will therefore address the problem of the right choice of case variables and present a method based on the analysis of unsatisfiable cores to guide this bounding process in a clever way. This will allow us to prove unsatisfiability in many cases and attenuate the incompleteness issue of the second kind.

3.4.1. Unsatisfiability and learning

Let $\mathbf{C} = \mathbf{L} \wedge \mathbf{N}$ be a pure conjunction of constraints. Such a constraints conjunction is always a special case of a CNF formula with a single literal per clause. The CNF property is invariant under any application of the presented linearization rules in Section 3.3. If the domains of all case variables in $\mathbf{C}$ are bounded, such that its nonlinear part $\mathbf{N}$ can be completely linearized to produce the formula $\mathbf{L_N}$, we will therefore get an equisatisfiable CNF formula $\mathbf{D} := \mathbf{L} \wedge \mathbf{L_N}$ in linear integer arithmetic. Recall the set theoretic notations for arbitrary CNF formulas from Subsection 1.1. The next Theorem relates the unsatisfiable cores of the linearization $\mathbf{D}$ to those of the original formula $\mathbf{C}$.

Theorem 3.1 Let the input $\mathbf{C}$ and hence its linearization $\mathbf{D}$ be unsatisfiable. If $\mathbf{U_D}$ is an unsatisfiable core for $\mathbf{D}$, then $\mathbf{U_C} := (\mathbf{U_D} \cap \mathbf{L}) \wedge \mathbf{N}$ is an unsatisfiable core for $\mathbf{C}$.

Proof. By the definition of an unsatisfiable core, we have $\mathbf{U_D} \subseteq \mathbf{D}$, which gives

$$\mathbf{U_D} = \mathbf{U_D} \cap \mathbf{D} = \mathbf{U_D} \cap (\mathbf{L} \wedge \mathbf{L_N}) = (\mathbf{U_D} \cap \mathbf{L}) \wedge (\mathbf{U_D} \cap \mathbf{L_N}) \subseteq (\mathbf{U_D} \cap \mathbf{L}) \wedge \mathbf{L_N}.$$

The rightmost CNF formula is equisatisfiable to $(\mathbf{U_D} \cap \mathbf{L}) \wedge \mathbf{N} = \mathbf{U_C}$. Altogether, this proves that $\mathbf{U_C}$ must contain a subformula that is equisatisfiable to the unsatisfiable core $\mathbf{U_D}$. Since further $\mathbf{U_C} \subseteq \mathbf{C}$ holds, it follows that $\mathbf{U_C}$ is an unsatisfiable core of $\mathbf{C}$. □

For many input formulas $\mathbf{C}$, the so far claimed boundedness for all variable domains that are used for case distinctions during the linearization process is not fulfilled. In this case, we need to introduce a conjunction of additional bounding constraints $\mathbf{B}$ and consider the input CNF formula $\mathbf{C}' := \mathbf{B} \wedge \mathbf{C}$ instead. This makes our method incomplete, since only `Sat` answers for $\mathbf{C}'$ imply the satisfiability of the original input $\mathbf{C}$. A first strategy to choose the newly added bounds in $\mathbf{B}$ as large as possible is foredoomed, as it easily produces a too hard problem for the internal linear arithmetic solver even if the logarithmic encoding of variable domains from Subsection 3.3.2 is used. An alternative idea is to start with bounds that make the domains small and enlarge them incrementally if necessary. Instead of enlarging all added bounds, we can further analyze the unsatisfiable core of the linearization to identify the bounds that need to be adapted. The core of this approach is the following refinement of Theorem 3.1 in the presence of bounding constraints.

Corollary 3.2 Let $\mathbf{B}$ be a conjunction of bounding constraints such that $\mathbf{C}' := \mathbf{B} \wedge \mathbf{C}$ can be linearized into the CNF formula $\mathbf{D}'$. If $\mathbf{U}_{\mathbf{D}'}$ is an unsatisfiable core for $\mathbf{D}'$ with $\mathbf{U}_{\mathbf{D}'} \cap \mathbf{B} = \emptyset$, then $\mathbf{U}_{\mathbf{C}} := (\mathbf{U}_{\mathbf{D}'} \cap \mathbf{L}) \wedge \mathbf{N}$ is an unsatisfiable core for the original input $\mathbf{C}$.

Proof. The purification of $\mathbf{C}$ is given by the decomposition $\mathbf{C} = \mathbf{L} \wedge \mathbf{N}$ into its linear and nonlinear parts $\mathbf{L}$ and $\mathbf{N}$, respectively. Since the bounding constraints in $\mathbf{B}$ are linear by definition, the corresponding purification of $\mathbf{C}'$ is given by $\mathbf{C}' = \mathbf{L}' \wedge \mathbf{N}'$ with

$$\mathbf{L}' := \mathbf{B} \wedge \mathbf{L} \qquad \text{and} \qquad \mathbf{N}' := \mathbf{N}.$$

If we apply Theorem 3.1 on $\mathbf{C}'$ instead of $\mathbf{C}$ and insert the given premise $\mathbf{U}_{\mathbf{D}'} \cap \mathbf{B} = \emptyset$, we obtain the unsatisfiable core for $\mathbf{C}'$ given by

$$\mathbf{U}_{\mathbf{C}'} := (\mathbf{U}_{\mathbf{D}'} \cap \mathbf{L}') \wedge \mathbf{N}' = (\mathbf{U}_{\mathbf{D}'} \cap (\mathbf{B} \wedge \mathbf{L})) \wedge \mathbf{N} = (\mathbf{U}_{\mathbf{D}'} \cap \mathbf{L}) \wedge \mathbf{N}.$$

From the rightmost representation of $\mathbf{U}_{\mathbf{C}'}$ we can easily see the relation $\mathbf{U}_{\mathbf{C}'} \subseteq \mathbf{L} \wedge \mathbf{N} = \mathbf{C}$. Hence, $\mathbf{U}_{\mathbf{C}} := \mathbf{U}_{\mathbf{C}'}$ is also an unsatisfiable core of $\mathbf{C}$. □

The benefit of Corollary 3.2 for our `CSplitModule` is twofold:

(i) If the unsatisifable core $\mathbf{U}_{\mathbf{D}'}$ of the linearization $\mathbf{D}'$ does not contain any of the auxiliary bounds in $\mathbf{B}$, we can deduce the unsatisfiability of the original formula $\mathbf{C} = \mathbf{L} \wedge \mathbf{N}$ and generate a corresponding unsatisfiable core $\mathbf{U}_{\mathbf{C}} := (\mathbf{U}_{\mathbf{D}'} \cap \mathbf{L}) \wedge \mathbf{N}$.

(ii) If, on the other hand, the unsatisfiable core $\mathbf{U}_{\mathbf{D}'}$ has a non-empty intersection with the auxiliary bounds in $\mathbf{B}$, then the constraints in $\mathbf{U}_{\mathbf{D}'} \cap \mathbf{B}$ are the *candidate bounds* that need to be enlarged before the next invocation of the linear arithmetic solver.

Unfortunately, the incremental enlargement of bounds in (ii) does not terminate for many input formulas after a finite number of iterations by reaching case (i). Our experiments showed that the initial choice of reduction sequences has the greatest impact on the termination of the algorithm in case of unsatisfiable input formulas. We will therefore look into this problem in more detail in the next Subsection 3.4.2. Here, we give a brief summary of additional implementation details of our `CSplitModule` that lead to a slight performance increase for both, satisfiability and unsatisfiability answers:

- As a first obvious strategy to ensure termination, we limit the maximal number of bounds refinement iterations, before the consistency check gets finally aborted with an `Unknown` result. In a single iteration, a subset of candidate bounds in $\mathbf{U}_{\mathbf{D}'} \cap \mathbf{B}$ gets bloated and the linear arithmetic solver is called on the adapted linearization. This step is very time critical and needs to be implemented efficiently. Let v be a variable whose domain is changed from D_v^{old} to D_v^{new}. An inspection of the $\Phi_{\texttt{Monomial}}$ formulas from Section 3.3 in which v is involved as the case variable shows that for

  – small domains we only need to add the case distinction clauses for the values in $D_v^{\text{new}} \setminus D_v^{\text{old}}$ and modify the bounding constraints accordingly.

  – for large domains the majority of $\Phi_{\texttt{Expansion}}$ subformulas stay completely untouched and only the bounds in the $\Phi_{\texttt{Digit}}$ subformulas need to be adapted.

  We implemented a recursive algorithm that simultaneously calculates the expansions in a positional numeral system to the base B for D_v^{old} and D_v^{new} and modifies precisely the changed clauses to transform the resulting linearization into a consistent state.

- For some linearizations, the internal linear arithmetic solver terminates quickly even for variable domain sizes in the order of millions, for others, variable domain sizes of five or less are already time critical. We therefore start for all variables v with an initial interval $D_v = [0, 1]$ and bloat the candidate domains in two phases:

  (i) In the first phase, the domains are enlarged linearly in both directions with a step size of one until a certain threshold is reached.

  (ii) If all candidates have exceeded the threshold size, we start an exponential bloating of the domains that are used for case-splits and activate the maximal domain for variables that are not involved in the case analysis.

  Furthermore, we limit the maximal number of candidates that are bloated in a single iteration, since we observed input formulas with 400 variables and more. From all potential bloating candidates in $\mathbf{U}_{\mathbf{D}'} \cap \mathbf{B}$, we prefer those with the smallest domains.

All of the mentioned parameters that control the behaviour of our `CSplitModule` are not hard-coded into it but can be set centrally in its corresponding settings file. We will report the best configuration that we found with the help of a sparse grid search in Section 3.5.

3.4.2. Optimal choice of reduction sequences

So far we have only considered the final linearization step for binary monomial equations of the form $x = v \cdot w$ for bounded and unbounded domains D_v. These binary monomial equations were produced in Subsection 3.2.2 from monomial equations $y_{\mathbf{p}} = \mathbf{x}^{\mathbf{p}}$ of arbitrary total degree $d_{\mathbf{p}} = \|\mathbf{p}\|_1 > 2$ by the choice of reduction sequences $\mathbf{b}_{\mathbf{p}} = (b_{\mathbf{p},1}, \ldots, b_{\mathbf{p},d_{\mathbf{p}}})$. These sequences define the order in which the variables are removed from the nonlinear monomial $\mathbf{x}^{\mathbf{p}}$. When multiple monomial equations $y_{\mathbf{p}_1} = \mathbf{x}^{\mathbf{p}_1}, \ldots, y_{\mathbf{p}_k} = \mathbf{x}^{\mathbf{p}_k}$ are involved, the choice of reduction sequences cannot be considered in isolation anymore. The interdependency of the reduction sequences $\mathbf{b}_{\mathbf{p}_1}, \ldots, \mathbf{b}_{\mathbf{p}_k}$ has a vast impact on the number of intermediate variables and thus the number of clauses in the final linearization.

Example 3.3 Consider the system of monomial equations $y_{\mathbf{p}_i} = \mathbf{x}^{\mathbf{p}_i}$ for $i = 1, \ldots, 3$ with

$$\mathbf{p}_1 = (1, 2, 2, 0), \ \mathbf{p}_2 = (2, 0, 2, 1), \ \mathbf{p}_3 = (1, 0, 2, 2).$$

Figure 3.1 shows the *reduction trees* for two different sets of reduction sequences that lead to a different number of intermediate nonlinear monomial equations:

(a) $\mathbf{b}_{\mathbf{p}_1} = (1, 2, 2, 3, 3)$, $\mathbf{b}_{\mathbf{p}_2} = (1, 1, 3, 4, 4)$, $\mathbf{b}_{\mathbf{p}_3} = (1, 3, 3, 4, 4)$ with 12 equations.

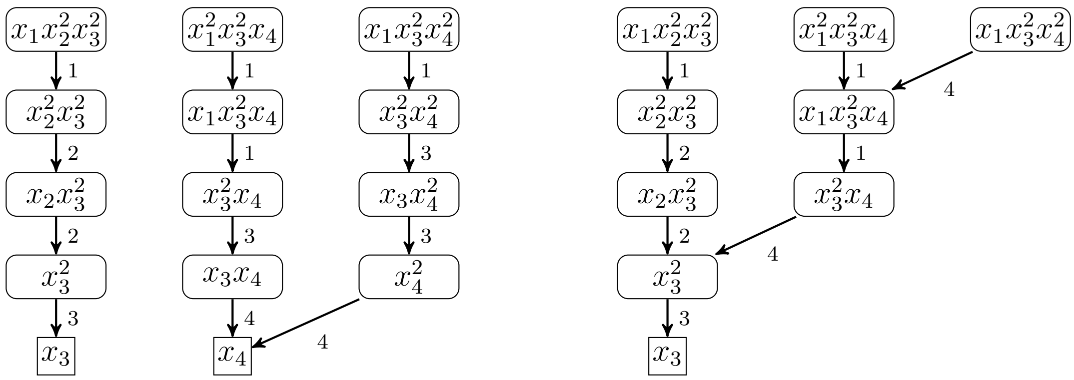
(b) $\mathbf{b}_{\mathbf{p}_1} = (1, 2, 2, 3, 3)$, $\mathbf{b}_{\mathbf{p}_2} = (1, 1, 4, 3, 3)$, $\mathbf{b}_{\mathbf{p}_3} = (4, 1, 4, 3, 3)$ with 8 equations.

The most desirable set of reduction sequences is the one that minimizes the number of intermediate nonlinear monomials in the reduction tree. It is easy to see that this minimization problem is NP-complete and thus too expensive as a subproblem of our linearization algorithm. In [Bor+12] this problem discussion closes with a reference to a "greedy approximation algorithm" without any information on the implementation details. After numerous experiments, we chose the following method as a tradeoff between the desired minimal cardinality of intermediate monomials and the avoidance of case variables with unbounded domains, since the later are the reason for a lack of termination. Let the set of exponents $E_0 := \{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ be sorted ascendingly in degree lexicographic order. Perform the following steps for $i = 1, \ldots, k$:

(i) In the monomial $\mathbf{x}^{\mathbf{p_i}}$ we have exactly one free choice of a variable that will not be used for case-splits during the linearization. We therefore select the last index $b_{d_{\mathbf{p}_i}}$ of the reduction sequence corresponding to the variable $x_{b_{d_{\mathbf{p}_i}}}$ with the largest domain.

(ii) Among all exponents $\mathbf{p} \in E_{i-1}$, choose the one with maximal degree $d_{\mathbf{p}}$ such that it

- is componentwise smaller than p_i and hence $\mathbf{x}^{\mathbf{p}_i}$ is reducible to $\mathbf{x}^{\mathbf{p}}$.
- contains the variable $x_{b_{d_{\mathbf{p}_i}}}$. By induction, it follows that $b_{d_{\mathbf{p}}} = b_{d_{\mathbf{p}_i}}$ and during the reduction process of $\mathbf{x}^{\mathbf{p}}$ the variable $x_{b_{d_{\mathbf{p}_i}}}$ was successfully avoided.



(a) (b)

Figure 3.1.: Reduction trees for two sets of reduction sequences $\mathbf{b}_{\mathbf{p}_1}, \mathbf{b}_{\mathbf{p}_2}, \mathbf{b}_{\mathbf{p}_3}$.

(iii) Construct the reduction sequence $\mathbf{b}_{\mathbf{p}_i}$ that reduces $\mathbf{p}_i$ to $\mathbf{p}$ and then follows the reduction sequence $\mathbf{b}_{\mathbf{p}}$. Add all used intermediate exponents to E_{i-1} to get E_i.

3.5. Benchmarking results and conclusion

We tested our `CSplitModule` on the QF_NIA division of the `SMT-LIB` benchmarking library [BFT16] and report our results in Table 3.1. The experiments were performed on a 2.7 GHz Intel Core i7-4800MQ CPU with a timeout of 60 seconds and 4 GB memory per benchmark. In [Bor+12], only the families AProVE, calypto and leipzig are considered with a timeout limit of 1200 seconds to conclude the supremacy of the case-splitting method. But our results clearly indicate that these three families were cherry-picked as they are the only with a reasonable performance. The strategies that we tested are:

CSplit only: `SATModule`→`CSplitModule`

Backends only: `SATModule`→`LRAModule`→`VSModule`→`CADModule`

CSplit + Backends: `SATModule`→`CSplitModule`→`LRAModule`→`VSModule`→`CADModule`

To find the best parameter combination, we performed a grid search on a validation subset of 500 benchmarks and finally picked the following setting:

- We choose $T = 32$ as the threshold between small and large domain sizes and also $B = 32$ as the base for the logarithmic encoding of large domains.
- The maximum number of bounds refinements is limited to 50 iterations. In every iteration, we choose at most three candidates whose bounds get enlarged. Candidates with a domain size of 300 or more get discarded and not considered for bloating.

- All variable domains are initially restricted to an interval of size one near to the zero point as many input formulas seem to have solutions near to the origin. They are bloated linearly with a step size of one until the threshold of three is reached. Afterwards, the exponential bloating routine starts.

It is worth mentioning that we were able to find parameter sets that gave better results when we restricted ourselves to one of the three above mentioned benchmark families. In calypto, the linear arithmetic solver terminates quickly independent from the domain sizes. Hence, it is advisible to select a much higher number of iterations and entirely remove the limit size of 300 for candidate domains. This behaviour is perpendicular to the leipzig family, where a fast rejection with an `Unknown` answer gives better results in later consistency checks of the outer DPLL loop.

QF_NIA Benchmarks		**CSplit only**		**CSplit + Backends**		**Backends only**	
Benchmark Family	Answer	Num	Avg	Num	Avg	Num	Avg
AProVE (2409)	`Sat` (1663)	894	1451	890	1392	627	893
	`Unsat` (320)	2	14	4	5960	49	647
	`Unknown` (426)	5	12898	0	0	0	0
	`Resout`	1508	59995	1515	59993	1503	59999
calypto (177)	`Sat` (80)	44	5036	45	4797	24	810
	`Unsat` (97)	6	28	9	1147	13	18
	`Unknown` (0)	21	21859	17	11798	124	167
	`Resout`	106	59986	106	60004	16	59989
CInteger (1818)	`Sat` (858)	10	27391	10	24151	0	0
	`Unsat` (150)	0	0	0	0	5	7294
	`Unknown` (810)	0	0	0	0	0	0
	`Resout`	1808	59991	1808	59993	1813	59966
ITS (17046)	`Sat` (9473)	32	17754	33	19957	8	5062
	`Unsat` (2360)	0	0	5	12160	46	2116
	`Unknown` (5213)	0	0	0	0	0	0
	`Resout`	17014	59997	17008	59989	16992	59991
LassoRanker (106)	`Sat` (4)	4	4568	3	50	3	165
	`Unsat` (100)	0	0	15	244	15	27
	`Unknown` (2)	59	3646	0	0	0	0
	`Resout`	43	59992	88	60002	88	59984

leipzig (167)	`Sat` (162)	73	5902	74	5889	15	7693
	`Unsat` (5)	0	0	0	0	0	0
	`Unknown` (0)	0	0	0	0	0	0
	`Resout`	94	59990	93	59986	152	59991
mcm (186)	`Sat` (25)	0	0	0	0	1	43761
	`Unsat` (0)	0	0	0	0	0	0
	`Unknown` (161)	0	0	0	0	0	0
	`Resout`	186	59983	186	59992	185	59989
SAT14 (1926)	`Sat` (1853)	11	6068	10	3180	11	3929
	`Unsat` (63)	0	0	0	0	11	16337
	`Unknown` (10)	0	0	0	0	0	0
	`Resout`	1915	59995	1916	59997	1904	60001
Ultimate Automizer/ LassoRanker (39)	`Sat` (6)	5	589	6	944	6	209
	`Unsat` (33)	7	8214	25	10891	27	7762
	`Unknown` (0)	1	2828	0	0	0	0
	`Resout`	26	59991	8	59937	6	59918
Total (23874)	`Sat` (14124)	1073	2684	1071	2641	778	1071
	`Unsat` (3128)	15	3846	58	6395	172	3232
	`Unknown` (6622)	86	8622	17	11798	124	167
	`Resout`	22700	59996	22728	59991	22800	59991

Table 3.1.: Benchmarking results of the `CSplitModule` on the QF_NIA division of the `SMT-LIB`.

Appendix A.

STropModule source code

Listing A.1: STropSettings.h

```
/**
 * @file STropSettings.h
 * @author Ömer Sali <oemer.sali@rwth-aachen.de>
 *
 * @version 2018-04-04
 * Created on 2017-09-13.
 */

#pragma once

#include "../../solver/ModuleSettings.h"
#include "../../solver/Manager.h"
#include "../SATModule/SATModule.h"
#include "../LRAModule/LRAModule.h"

namespace smtrat
{
 enum class SeparatorType {STRICT = 0, WEAK = 1};

 struct STropSettings1
 {
  /// Name of the Module
  static constexpr auto moduleName = "STropModule<STropSettings1>";
  /// Type of linear separating hyperplane to search for
  static constexpr SeparatorType separatorType = SeparatorType::STRICT;
  /// Linear real arithmetic solver to call for the linearized formula
  struct LRASolver : public Manager
  {
   LRASolver() : Manager()
   {
    setStrategy({
     addBackend<SATModule<SATSettings1>>({
      addBackend<LRAModule<LRASettings1>>()
     })
    });
   }
  };
 };
}
```

Listing A.2: STropModule.h

```
/**
 * @file STropModule.h
 * @author Ömer Sali <oemer.sali@rwth-aachen.de>
 *
 * @version 2018-04-04
 * Created on 2017-09-13.
```

```
 */

#pragma once

#include "../../solver/Module.h"
#include "STropStatistics.h"
#include "STropSettings.h"

namespace smtrat
{
 template<typename Settings>
 class STropModule : public Module
 {
  private:
#ifdef SMTRAT_DEVOPTION_Statistics
   STropStatistics mStatistics;
#endif
   /**
    * Represents the normal vector component and the sign variable
    * assigned to a variable in an original constraint.
    */
   struct Moment
   {
    /// Normal vector component of the separating hyperplane
    const carl::Variable mNormalVector;
    /// Boolean variable representing the sign variant
    const carl::Variable mSignVariant;
    /// Flag that indicates whether this moment is used for
        linearization
    bool mUsed;

    Moment()
     : mNormalVector(carl::freshRealVariable())
     , mSignVariant(carl::freshBooleanVariable())
     , mUsed(false)
    {}
   };

   /// Maps a variable to the components of the moment function
   std::unordered_map<carl::Variable, Moment> mMoments;

   /**
    * Represents a term of an original constraint and assigns
    * him a rating variable if a weak separator is searched.
    */
   struct Vertex
   {
    /// Coefficient of the assigned term
    const Rational mCoefficient;
    /// Monomial of the assigned term
    const carl::Monomial::Arg mMonomial;
    /// Rating variable of the term for a weak separator
    const carl::Variable mRating;

    Vertex(const TermT& term)
     : mCoefficient(term.coeff())
     , mMonomial(term.monomial())
     , mRating(
      Settings::separatorType == SeparatorType::WEAK ?
      carl::freshRealVariable() : carl::Variable::NO_VARIABLE)
    {}
   };

   /// Subdivides the relations into classes with the same linearization
        result
   enum class Direction {NONE = 0, BOTH = 0, NEGATIVE = 1, POSITIVE =
       2};

```

```
    /**
     * Represents the class of all original constraints with the same
     * left hand side after a normalization. Here, the set of all received
     * relations of constraints with the same left hand side is stored. At any
     * one time only one relation can be active and used for linearization.
     */
    struct Separator
    {
      /// Bias variable of the separating hyperplane
      const carl::Variable mBias;
      /// Vertices for all terms of the normalized left hand side
      const std::vector<Vertex> mVertices;
      /// Relations of constraints with the same left hand side
      std::set<carl::Relation> mRelations;
      /// Direction currently used for linearization
      Direction mActiveDirection;

      Separator(const Poly& normalization)
        : mBias(carl::freshRealVariable())
        , mVertices(normalization.begin(), normalization.end())
        , mRelations()
        , mActiveDirection(Direction::NONE)
      {}
    };

    /// Maps a normalized left hand side of a constraint to its separator
    std::unordered_map<Poly, Separator> mSeparators;
    /// Stores the Separators that were updated since the last check call
    std::unordered_set<Separator *> mChangedSeparators;
    /// Counts the number of relation pairs that prohibit an application of this method
    size_t mRelationalConflicts;
    /// Stores the sets of separators that were found to be undecidable by the LRA solver
    typedef std::vector<std::pair<const Separator *, const Direction>> Conflict;
    std::vector<Conflict> mLinearizationConflicts;
    /// Stores whether the last consistency check was done by the backends
    bool mCheckedWithBackends;
    /// Handle to the linear real arithmetic solver
    typename Settings::LRASolver mLRASolver;

  public:
    typedef Settings SettingsType;

    std::string moduleName() const
    {
      return SettingsType::moduleName;
    }

    STropModule(const ModuleInput* _formula, RuntimeSettings* _settings, Conditionals& _conditionals, Manager* _manager = nullptr);

    /**
     * The module has to take the given sub-formula of the received formula into account.
     * @param _subformula The sub-formula to take additionally into account.
     * @return False, if it can be easily decided that this sub-formula causes a conflict with
     *         the already considered sub-formulas;
     *         True, otherwise.
     */
    bool addCore(ModuleInput::const_iterator _subformula);
```

```

    /**
     * Removes the subformula of the received formula at the given
        position to the considered ones of this module.
     * Note that this includes every stored calculation which depended on
         this subformula, but should keep the other
     * stored calculation, if possible, untouched.
     * @param _subformula The position of the subformula to remove.
     */
    void removeCore(ModuleInput::const_iterator _subformula);

    /**
     * Updates the current assignment into the model.
     * Note, that this is a unique but possibly symbolic assignment maybe
         containing newly introduced variables.
     */
    void updateModel() const;

    /**
     * Checks the received formula for consistency.
     * @return SAT,  if the received formula is satisfiable;
     *     UNSAT,   if the received formula is not satisfiable;
     *     UNKNOWN, otherwise.
     */
    Answer checkCore();

  private:
    /**
     * Creates the linearization for the given separator with the active
        relation.
     * @param separator The separator object that stores the construction
         information.
     * @return Formula that is satisfiable iff such a separating
        hyperplane exists.
     */
    inline FormulaT createLinearization(const Separator& separator);

    /**
     * Creates the formula for the hyperplane that linearly separates at
        least one
     * variant positive frame vertex from all variant negative frame
        vertices. If a
     * weak separator is searched, the corresponding rating is included.
     * @param separator The separator object that stores the construction
         information.
     * @param negated True, if the formula for the negated polynomial
        shall be constructed.
     *       False, if the formula for the original polynomial shall be
        constructed.
     * @return Formula that is satisfiable iff such a separating
        hyperplane exists.
     */
    FormulaT createSeparator(const Separator& separator, bool negated);

    /**
     * Asserts/Removes the given formula to/from the LRA solver.
     * @param formula The formula to assert/remove to the LRA solver.
     * @param assert True, if formula shall be asserted;
     *      False, if formula shall be removed.
     */
    inline void propagateFormula(const FormulaT& formula, bool assert);
  };
}
```

Listing A.3: STropModule.cpp

```
/**
```

```
 * @file STropModule.cpp
 * @author Ömer Sali <oemer.sali@rwth-aachen.de>
 *
 * @version 2018-04-04
 * Created on 2017-09-13.
 */

#include "STropModule.h"

namespace smtrat
{
 template<class Settings>
 STropModule<Settings>::STropModule(const ModuleInput* _formula, RuntimeSettings*, Conditionals& _conditionals, Manager* _manager)
  : Module(_formula, _conditionals, _manager)
  , mMoments()
  , mSeparators()
  , mChangedSeparators()
  , mRelationalConflicts(0)
  , mLinearizationConflicts()
  , mCheckedWithBackends(false)
#ifdef SMTRAT_DEVOPTION_Statistics
  , mStatistics(Settings::moduleName)
#endif
 {}

 template<class Settings>
 bool STropModule<Settings>::addCore(ModuleInput::const_iterator _subformula)
 {
  addReceivedSubformulaToPassedFormula(_subformula);
  const FormulaT& formula{_subformula->formula()};
  if (formula.getType() == carl::FormulaType::FALSE)
   mInfeasibleSubsets.push_back({formula});
  else if (formula.getType() == carl::FormulaType::CONSTRAINT)
  {
   /// Normalize the left hand side of the constraint and turn the relation accordingly
   const ConstraintT& constraint{formula.constraint()};
   const Poly normalization{constraint.lhs().normalize()};
   carl::Relation relation{constraint.relation()};
   if (carl::isNegative(constraint.lhs().lcoeff()))
    relation = carl::turnAroundRelation(relation);

   /// Store the normalized constraint and mark the separator object as changed
   Separator& separator{mSeparators.emplace(normalization, normalization).first->second};
   separator.mRelations.insert(relation);
   mChangedSeparators.insert(&separator);

   /// Check if the asserted constraint prohibits the application of this method
   if (relation == carl::Relation::EQ
    || (relation == carl::Relation::LEQ
     && separator.mRelations.count(carl::Relation::GEQ))
    || (relation == carl::Relation::GEQ
     && separator.mRelations.count(carl::Relation::LEQ)))
    ++mRelationalConflicts;

   /// Check if the asserted relation trivially conflicts with other asserted relations
   switch (relation)
   {
    case carl::Relation::EQ:
     if (separator.mRelations.count(carl::Relation::NEQ))
      mInfeasibleSubsets.push_back({
       FormulaT(normalization, carl::Relation::EQ),
```

```
            FormulaT(normalization, carl::Relation::NEQ)
          });
        if (separator.mRelations.count(carl::Relation::LESS))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, carl::Relation::EQ),
            FormulaT(normalization, carl::Relation::LESS)
          });
        if (separator.mRelations.count(carl::Relation::GREATER))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, carl::Relation::EQ),
            FormulaT(normalization, carl::Relation::GREATER)
          });
        break;
      case carl::Relation::NEQ:
        if (separator.mRelations.count(carl::Relation::EQ))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, carl::Relation::NEQ),
            FormulaT(normalization, carl::Relation::EQ)
          });
        break;
      case carl::Relation::LESS:
        if (separator.mRelations.count(carl::Relation::EQ))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, carl::Relation::LESS),
            FormulaT(normalization, carl::Relation::EQ)
          });
        if (separator.mRelations.count(carl::Relation::GEQ))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, carl::Relation::LESS),
            FormulaT(normalization, carl::Relation::GEQ)
          });
      case carl::Relation::LEQ:
        if (separator.mRelations.count(carl::Relation::GREATER))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, relation),
            FormulaT(normalization, carl::Relation::GREATER)
          });
        break;
      case carl::Relation::GREATER:
        if (separator.mRelations.count(carl::Relation::EQ))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, carl::Relation::GREATER),
            FormulaT(normalization, carl::Relation::EQ)
          });
        if (separator.mRelations.count(carl::Relation::LEQ))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, carl::Relation::GREATER),
            FormulaT(normalization, carl::Relation::LEQ)
          });
      case carl::Relation::GEQ:
        if (separator.mRelations.count(carl::Relation::LESS))
          mInfeasibleSubsets.push_back({
            FormulaT(normalization, relation),
            FormulaT(normalization, carl::Relation::LESS)
          });
        break;
      default:
        assert(false);
    }
  }
  return mInfeasibleSubsets.empty();
}

template<class Settings>
void STropModule<Settings>::removeCore(ModuleInput::const_iterator _subformula)
{
```

```
  const FormulaT& formula{_subformula->formula()};
  if (formula.getType() == carl::FormulaType::CONSTRAINT)
  {
   /// Normalize the left hand side of the constraint and turn the relation accordingly
   const ConstraintT& constraint{formula.constraint()};
   const Poly normalization{constraint.lhs().normalize()};
   carl::Relation relation{constraint.relation()};
   if (carl::isNegative(constraint.lhs().lcoeff()))
    relation = carl::turnAroundRelation(relation);

   /// Retrieve the normalized constraint and mark the separator object as changed
   Separator& separator{mSeparators.at(normalization)};
   separator.mRelations.erase(relation);
   mChangedSeparators.insert(&separator);

   /// Check if the removed constraint prohibited the application of this method
   if (relation == carl::Relation::EQ
    || (relation == carl::Relation::LEQ
     && separator.mRelations.count(carl::Relation::GEQ))
    || (relation == carl::Relation::GEQ
     && separator.mRelations.count(carl::Relation::LEQ)))
    --mRelationalConflicts;
  }
 }

 template<class Settings>
 void STropModule<Settings>::updateModel() const
 {
  if (!mModelComputed)
  {
   if (mCheckedWithBackends)
   {
    clearModel();
    getBackendsModel();
    excludeNotReceivedVariablesFromModel();
   }
   else
   {
    /// Stores all informations retrieved from the LRA solver to construct the model
    struct Weight
    {
     const carl::Variable& mVariable;
     Rational mExponent;
     const bool mSign;

     Weight(const carl::Variable& variable, const Rational& exponent, const bool sign)
      : mVariable(variable)
      , mExponent(exponent)
      , mSign(sign)
     {}
    };
    std::vector<Weight> weights;

    /// Retrieve the sign and exponent for every active variable
    const Model& LRAModel{mLRASolver.model()};
    Rational gcd(0);
    for (const auto& momentsEntry : mMoments)
    {
     const carl::Variable& variable{momentsEntry.first};
     const Moment& moment{momentsEntry.second};
     if (moment.mUsed)
     {
      auto signIter{LRAModel.find(moment.mSignVariant)};
```

```
        weights.emplace_back(
         variable,
         LRAModel.at(moment.mNormalVector).asRational(),
         signIter != LRAModel.end() && signIter->second.asBool()
        );
        carl::gcd_assign(gcd, weights.back().mExponent);
       }
      }

      /// Calculate smallest possible integer valued exponents
      if (gcd != ZERO_RATIONAL && gcd != ONE_RATIONAL)
       for (Weight& weight : weights)
        weight.mExponent /= gcd;

      /// Find model by increasingly testing the sample base
      Rational base{ZERO_RATIONAL};
      do
      {
       ++base;
       clearModel();
       for (const Weight& weight : weights)
       {
        Rational value{carl::isNegative(weight.mExponent) ? carl::
            reciprocal(base) : base};
        carl::pow_assign(value, carl::toInt<carl::uint>(carl::abs(weight.
            mExponent)));
        if (weight.mSign)
         value *= MINUS_ONE_RATIONAL;
        mModel.emplace(weight.mVariable, value);
       }
      }
      while (!rReceivedFormula().satisfiedBy(mModel));
     }
     mModelComputed = true;
    }
   }

   template<class Settings>
   Answer STropModule<Settings>::checkCore()
   {
    /// Report unsatisfiability if the already found conflicts are still
        unresolved
    if (!mInfeasibleSubsets.empty())
     return Answer::UNSAT;

    /// Predicate that decides if the given conflict is a subset of the
        asserted constraints
    const auto hasConflict = [&](const Conflict& conflict) {
     return std::all_of(
      conflict.begin(),
      conflict.end(),
      [&](const auto& conflictEntry) {
       return ((conflictEntry.second == Direction::NEGATIVE
         || conflictEntry.second == Direction::BOTH)
         && (conflictEntry.first->mRelations.count(carl::Relation::LESS)
          || conflictEntry.first->mRelations.count(carl::Relation::LEQ)))
         || ((conflictEntry.second == Direction::POSITIVE
          || conflictEntry.second == Direction::BOTH)
          && (conflictEntry.first->mRelations.count(carl::Relation::
             GREATER)
           || conflictEntry.first->mRelations.count(carl::Relation::GEQ)))
         || (conflictEntry.second == Direction::BOTH
          && conflictEntry.first->mRelations.count(carl::Relation::NEQ));
      }
     );
    };

```

```
  /// Apply the method only if the asserted formula is not trivially undecidable
  if (!mRelationalConflicts
   && rReceivedFormula().isConstraintConjunction()
   && std::none_of(mLinearizationConflicts.begin(), mLinearizationConflicts.end(), hasConflict))
  {
   /// Update the linearization of all changed separators
   for (Separator *separatorPtr : mChangedSeparators)
   {
    Separator& separator{*separatorPtr};

    /// Determine the direction that shall be active
    Direction direction;
    if (separator.mRelations.empty())
     direction = Direction::NONE;
    else if ((separator.mActiveDirection == Direction::NEGATIVE
      && ((separator.mRelations.count(carl::Relation::LESS)
       || separator.mRelations.count(carl::Relation::LEQ))))
     || (separator.mActiveDirection == Direction::POSITIVE
      && ((separator.mRelations.count(carl::Relation::GREATER)
       || separator.mRelations.count(carl::Relation::GEQ)))))
     direction = separator.mActiveDirection;
    else
     switch (*separator.mRelations.rbegin())
     {
      case carl::Relation::NEQ:
       direction = Direction::BOTH;
       break;
      case carl::Relation::LESS:
      case carl::Relation::LEQ:
       direction = Direction::NEGATIVE;
       break;
      case carl::Relation::GREATER:
      case carl::Relation::GEQ:
       direction = Direction::POSITIVE;
       break;
      default:
       assert(false);
     }

    /// Update the linearization if the direction has changed
    if (separator.mActiveDirection != direction)
    {
     if (separator.mActiveDirection != Direction::NONE)
      propagateFormula(createLinearization(separator), false);
     separator.mActiveDirection = direction;
     if (separator.mActiveDirection != Direction::NONE)
      propagateFormula(createLinearization(separator), true);
    }
   }
   mChangedSeparators.clear();

   /// Restrict the normal vector component of integral variables to positive values
   for (auto& momentsEntry : mMoments)
   {
    const carl::Variable& variable{momentsEntry.first};
    Moment& moment{momentsEntry.second};
    if (variable.type() == carl::VariableType::VT_INT
     && moment.mUsed != receivedVariable(variable))
    {
     moment.mUsed = !moment.mUsed;
     propagateFormula(FormulaT(Poly(moment.mNormalVector), carl::Relation::GEQ), moment.mUsed);
    }
   }

```

```
    /// Check the constructed linearization with the LRA solver
    switch (mLRASolver.check(true))
    {
      case Answer::SAT:
        mCheckedWithBackends = false;
        return Answer::SAT;
      case Answer::UNSAT:
        /// Learn the conflicting set of separators to avoid its recheck in the future
        const std::vector<FormulaSetT> LRAConflicts{mLRASolver.infeasibleSubsets()};
        for (const FormulaSetT& LRAConflict : LRAConflicts)
        {
          carl::Variables variables;
          for (const FormulaT& formula : LRAConflict)
            formula.allVars(variables);
          Conflict conflict;
          for (const auto& separatorsEntry : mSeparators)
          {
            const Separator& separator{separatorsEntry.second};
            if (separator.mActiveDirection != Direction::NONE
              && variables.count(separator.mBias))
              conflict.emplace_back(&separator, separator.mActiveDirection);
          }
          mLinearizationConflicts.emplace_back(std::move(conflict));
        }
    }
  }

  /// Check the asserted formula with the backends
  mCheckedWithBackends = true;
  Answer answer{runBackends()};
  if (answer == Answer::UNSAT)
    getInfeasibleSubsets();
  return answer;
}

template<class Settings>
inline FormulaT STropModule<Settings>::createLinearization(const Separator& separator)
{
  switch (separator.mActiveDirection)
  {
    case Direction::POSITIVE:
      return createSeparator(separator, false);
    case Direction::NEGATIVE:
      return createSeparator(separator, true);
    case Direction::BOTH:
      return FormulaT(
        carl::FormulaType::XOR,
        createSeparator(separator, false),
        createSeparator(separator, true)
      );
    default:
      assert(false);
  }
}

template<class Settings>
FormulaT STropModule<Settings>::createSeparator(const Separator& separator, bool negated)
{
  Poly totalRating;
  FormulasT disjunctions, conjunctions;
  for (const Vertex& vertex : separator.mVertices)
  {
    /// Create the hyperplane and sign change formula
    Poly hyperplane{separator.mBias};
```

```
    FormulaT signChangeFormula;
    if (vertex.mMonomial)
    {
      FormulasT signChangeSubformulas;
      for (const auto& exponent : vertex.mMonomial->exponents())
      {
        const auto& moment{mMoments[exponent.first]};
        hyperplane += Rational(exponent.second)*moment.mNormalVector;
        if (exponent.second%2)
          signChangeSubformulas.emplace_back(moment.mSignVariant);
      }
      signChangeFormula = FormulaT(carl::FormulaType::XOR, move(signChangeSubformulas));
    }

    /// Create the rating case distinction formula
    if (Settings::separatorType == SeparatorType::WEAK)
    {
      totalRating += vertex.mRating;
      conjunctions.emplace_back(
        carl::FormulaType::IMPLIES,
        FormulaT(hyperplane, carl::Relation::LESS),
        FormulaT(Poly(vertex.mRating), carl::Relation::EQ)
      );
      const Rational coefficient{negated ? -vertex.mCoefficient : vertex.mCoefficient};
      conjunctions.emplace_back(
        carl::FormulaType::IMPLIES,
        FormulaT(hyperplane, carl::Relation::EQ),
        FormulaT(
          carl::FormulaType::AND,
          FormulaT(
            carl::FormulaType::IMPLIES,
            signChangeFormula,
            FormulaT(vertex.mRating+coefficient, carl::Relation::EQ)
          ),
          FormulaT(
            carl::FormulaType::IMPLIES,
            signChangeFormula.negated(),
            FormulaT(vertex.mRating-coefficient, carl::Relation::EQ)
          )
        )
      );
    }

    /// Create the strict/weak linear saparating hyperplane
    bool positive{carl::isPositive(vertex.mCoefficient) != negated};
    disjunctions.emplace_back(
      FormulaT(
        carl::FormulaType::IMPLIES,
        positive ? signChangeFormula.negated() : signChangeFormula,
        FormulaT(hyperplane, Settings::separatorType == SeparatorType::STRICT ? carl::Relation::LEQ : carl::Relation::LESS)
      ).negated()
    );
    conjunctions.emplace_back(
      carl::FormulaType::IMPLIES,
      positive ? move(signChangeFormula) : move(signChangeFormula.negated()),
      FormulaT(move(hyperplane), carl::Relation::LEQ)
    );
  }
  if (Settings::separatorType == SeparatorType::WEAK)
    conjunctions.emplace_back(totalRating, carl::Relation::GREATER);
  return FormulaT(
    carl::FormulaType::AND,
    FormulaT(carl::FormulaType::OR, move(disjunctions)),
```

```
      FormulaT(carl::FormulaType::AND, move(conjunctions))
    );
  }

  template<class Settings>
  inline void STropModule<Settings>::propagateFormula(const FormulaT& formula, bool assert)
  {
    if (assert)
      mLRASolver.add(formula);
    else if (formula.getType() == carl::FormulaType::AND)
    {
      auto iter{mLRASolver.formulaBegin()};
      for (const auto& subformula : formula.subformulas())
        iter = mLRASolver.remove(std::find(iter, mLRASolver.formulaEnd(), subformula));
    }
    else
      mLRASolver.remove(std::find(mLRASolver.formulaBegin(), mLRASolver.formulaEnd(), formula));
  }
}

#include "Instantiation.h"
```

Appendix B.

CSplitModule source code

Listing B.1: Bimap.h

```
/**
 * @file CSplitModule.h
 * @author Ömer Sali <oemer.sali@rwth-aachen.de>
 *
 * @version 2018-04-04
 * Created on 2017-11-01.
 */

#pragma

#include <forward_list>
#include <set>

namespace smtrat
{
 /**
  * Container that stores expensive to construct objects and allows the
  * fast lookup with respect to two independent keys within the objects.
  */
 template<class Class, typename KeyType, KeyType Class::*FirstKey,
    KeyType Class::*SecondKey>
 class Bimap
 {
  public:
   typedef std::forward_list<Class> Data;
   typedef typename Data::iterator Iterator;
   typedef typename Data::const_iterator ConstIterator;

  private:
   /// Comparator that performs a heterogeneous lookup on the first key
   struct FirstCompare
   {
    using is_transparent = void;

    bool operator()(const Iterator& lhs, const Iterator& rhs) const
    {
     return (*lhs).*FirstKey<(*rhs).*FirstKey;
    }

    bool operator()(const Iterator& lhs, const KeyType& rhs) const
    {
     return (*lhs).*FirstKey<rhs;
    }

    bool operator()(const KeyType& lhs, const Iterator& rhs) const
    {
     return lhs<(*rhs).*FirstKey;
    }
   };

```

```
    /// Comparator that performs a heterogeneous lookup on the second key
    struct SecondCompare
    {
     using is_transparent = void;

     bool operator()(const Iterator& lhs, const Iterator& rhs) const
     {
      return (*lhs).*SecondKey<(*rhs).*SecondKey;
     }

     bool operator()(const Iterator& lhs, const KeyType& rhs) const
     {
      return (*lhs).*SecondKey<rhs;
     }

     bool operator()(const KeyType& lhs, const Iterator& rhs) const
     {
      return lhs<(*rhs).*SecondKey;
     }
    };

    Data mData;
    std::set<Iterator, FirstCompare> mFirstMap;
    std::set<Iterator, SecondCompare> mSecondMap;

   public:
    Iterator begin() noexcept
    {
     return mData.begin();
    }

    ConstIterator begin() const noexcept
    {
     return mData.begin();
    }

    Iterator end() noexcept
    {
     return mData.end();
    }

    ConstIterator end() const noexcept
    {
     return mData.end();
    }

    Class& firstAt(const KeyType& firstKey)
    {
     return *(*mFirstMap.find(firstKey));
    }

    Class& secondAt(const KeyType& secondKey)
    {
     return *(*mSecondMap.find(secondKey));
    }

    Iterator firstFind(const KeyType& firstKey)
    {
     auto firstIter{mFirstMap.find(firstKey)};
     if (firstIter == mFirstMap.end())
      return mData.end();
     else
      return *firstIter;
    }

    Iterator secondFind(const KeyType& secondKey)
    {
     auto secondIter{mSecondMap.find(secondKey)};
```

```
    if (secondIter == mSecondMap.end())
     return mData.end();
    else
     return *secondIter;
   }

   template<typename... Args>
   Iterator emplace(Args&&... args)
   {
    mData.emplace_front(std::move(args)...);
    mFirstMap.emplace(mData.begin());
    mSecondMap.emplace(mData.begin());
    return mData.begin();
   }
 };
}
```

Listing B.2: CSplitSettings.h

```
/**
 * @file CSplitSettings.h
 * @author Ömer Sali <oemer.sali@rwth-aachen.de>
 *
 * @version 2018-04-04
 * Created on 2017-11-01.
 */

#pragma

#include "../../solver/ModuleSettings.h"
#include "../../solver/Manager.h"
#include "../SATModule/SATModule.h"
#include "../LRAModule/LRAModule.h"

namespace smtrat
{
 struct CSplitSettings1
 {
  /// Name of the Module
  static constexpr auto moduleName = "CSplitModule<CSplitSettings1>";
  /// Limit size for the domain of variables that need to be expanded
  static constexpr size_t maxDomainSize = 32;
  /// Base number 2 <= expansionBase <= maxDomainSize for the expansion
  static constexpr size_t expansionBase = 32;
  /// Common denominator for the discretization of rational variables
  static constexpr size_t discrDenom = 16;
  /// Maximum number of iterations before returning unknown (0 =
      infinite)
  static constexpr size_t maxIter = 50;
  /// Radius of initial variable domains
  static constexpr size_t initialRadius = 1;
  /// Threshold radius to
  ///    - start exponential bloating of variables used for case splits
  ///    - activate full domains of variables not used for case splits
  static constexpr size_t thresholdRadius = 3;
  /// Maximal radius of domain that still gets bloated (0 = infinite)
  static constexpr size_t maximalRadius = 300;
  /// Maximal number of bounds to bloat in one iteration (0 = infinite)
  static constexpr size_t maxBloatedDomains = 3;
  /// Linear integer arithmetic module to call for the linearized
      formula
  struct LIASolver : public Manager
  {
   LIASolver() : Manager()
   {
    setStrategy({
     addBackend<SATModule<SATSettings1>>({
```

```
          addBackend<LRAModule<LRASettings1>>()
        })
      });
    }
  };
 };
}
```

Listing B.3: CSplitModule.h

```
/**
 * @file CSplitModule.h
 * @author Ömer Sali <oemer.sali@rwth-aachen.de>
 *
 * @version 2018-04-04
 * Created on 2017-11-01.
 */

#pragma once

#include "../../datastructures/VariableBounds.h"
#include "../../solver/Module.h"
#include "Bimap.h"
#include "CSplitStatistics.h"
#include "CSplitSettings.h"

namespace smtrat
{
 template<typename Settings>
 class CSplitModule : public Module
 {
  private:
#ifdef SMTRAT_DEVOPTION_Statistics
   CSplitStatistics mStatistics;
#endif
   /**
    * Represents the substitution variables of a nonlinear monomial
    * in a positional notation to the basis Settings::expansionBase.
    */
   struct Purification
   {
    /// Variable sequence used for the virtual positional notation
    std::vector<carl::Variable> mSubstitutions;
    /// Variable that is eliminated from the monomial during reduction
    carl::Variable mReduction;
    /// Number of active constraints in which the monomial is included
    size_t mUsage;
    /// Flag that indicates whether this purification is used for
        linearization
    bool mActive;

    Purification()
     : mSubstitutions()
     , mReduction(carl::Variable::NO_VARIABLE)
     , mUsage(0)
     , mActive(false)
    {
     mSubstitutions.emplace_back(carl::freshIntegerVariable());
    }
   };

   /// Maps a monomial to its purification information
   std::map<carl::Monomial::Arg, Purification> mPurifications;

   /// Subdivides the size of a variable domain into three classes:
   /// - SMALL,   if domain size <= Settings::maxDomainSize
   /// - LARGE,   if Settings::maxDomainSize < domain size < infinity
```

```
    /// - UNBOUNDED, if domain size = infinity
    enum class DomainSize{SMALL = 0, LARGE = 1, UNBOUNDED = 2};

    /**
     * Represents the quotients and remainders of a variable in
     * a positional notation to the basis Settings::expansionBase.
     */
    struct Expansion
    {
     /// Original variable to which this expansion is dedicated to and its discrete substitute
     const carl::Variable mRationalization, mDiscretization;
     /// Center point of the domain where the search starts
     Rational mNucleus;
     /// Size of the maximal domain
     DomainSize mMaximalDomainSize;
     /// Maximal domain deduced from received constraints and the currently active domain
     RationalInterval mMaximalDomain, mActiveDomain;
     /// Sequences of quotients and remainders used for the virtual positional notation
     std::vector<carl::Variable> mQuotients, mRemainders;
     /// Active purifications of monomials that contain the rationalization variable
     std::vector<Purification *> mPurifications;
     /// Flag that indicates whether the variable bounds changed since last check call
     bool mChangedBounds;

     Expansion(const carl::Variable& rationalization)
      : mRationalization(rationalization)
      , mDiscretization(rationalization.type() == carl::VariableType::VT_INT ? rationalization : carl::freshIntegerVariable())
      , mNucleus(ZERO_RATIONAL)
      , mMaximalDomainSize(DomainSize::UNBOUNDED)
      , mMaximalDomain(RationalInterval::unboundedInterval())
      , mActiveDomain(RationalInterval::emptyInterval())
      , mChangedBounds(false)
     {
      mQuotients.emplace_back(mDiscretization);
     }
    };

    Bimap<Expansion, const carl::Variable, &Expansion::mRationalization, &Expansion::mDiscretization> mExpansions;

    /**
     * Represents the class of all original constraints with the same
     * left hand side after a normalization. Here, the set of all received
     * relations of constraints with the same left hand side is stored.
     */
    struct Linearization
    {
     /// Normalization of the original constraint to which this linearization is dedicated to
     const Poly mNormalization, mLinearization;
     /// Flag that indicates a sign change between the leading coefficient of normalization and linearization
     const bool mParity;
     /// Purifications of the original nonlinear monomials
     const std::vector<Purification *> mPurifications;
     /// Flag that indicates whether the original constraint contains real variables
     const bool mHasRealVariables;
     /// Relations of constraints with the same left hand side
     std::unordered_set<carl::Relation> mRelations;

```

```
    Linearization(const Poly& normalization, const Poly& linearization,
        std::vector<Purification *>&& purifications, bool
        hasRealVariables)
      : mNormalization(normalization)
      , mLinearization(linearization.normalize())
      , mParity(carl::isNegative(linearization.lcoeff()))
      , mPurifications(std::move(purifications))
      , mHasRealVariables(std::move(hasRealVariables))
    {}
  };

  Bimap<Linearization, const Poly, &Linearization::mNormalization, &
      Linearization::mLinearization> mLinearizations;

  /// Helper class that extracts the variable domains
  vb::VariableBounds<FormulaT> mVariableBounds;
  /// Stores the last model for the linearization that was found by the
      LIA solver
  Model mLIAModel;
  /// Stores whether the last consistency check was done by the
      backends
  bool mCheckedWithBackends;
  /// Handle to the linear integer arithmetic module
  typename Settings::LIASolver mLIASolver;

 public:
  typedef Settings SettingsType;

  std::string moduleName() const
  {
    return SettingsType::moduleName;
  }

  CSplitModule(const ModuleInput* _formula, RuntimeSettings* _settings,
      Conditionals& _conditionals, Manager* _manager = nullptr);

  /**
   * The module has to take the given sub-formula of the received
       formula into account.
   * @param _subformula The sub-formula to take additionally into
       account.
   * @return False, if it can be easily decided that this sub-formula
       causes a conflict with
   *        the already considered sub-formulas;
   *        True, otherwise.
   */
  bool addCore( ModuleInput::const_iterator _subformula );

  /**
   * Removes the subformula of the received formula at the given
       position to the considered ones of this module.
   * Note that this includes every stored calculation which depended on
        this subformula, but should keep the other
   * stored calculation, if possible, untouched.
   * @param _subformula The position of the subformula to remove.
   */
  void removeCore( ModuleInput::const_iterator _subformula );

  /**
   * Updates the current assignment into the model.
   * Note, that this is a unique but possibly symbolic assignment maybe
        containing newly introduced variables.
   */
  void updateModel() const;

  /**
   * Checks the received formula for consistency.
   * @return SAT,  if the received formula is satisfiable;
```

```
	 *      UNSAT,   if the received formula is not satisfiable;
	 *      UNKNOWN, otherwise.
	 */
	Answer checkCore();

  private:
	/**
	 * Resets all expansions to the center points of the variable domains
	   and
	 * constructs a new tree of reductions for the currently active
	   monomials.
	 * @return True,  if there exists a maximal domain with no integral
	   points;
	 *       False, otherwise.
	 */
	bool resetExpansions();

	/**
	 * Bloats the active domains of a subset of variables that are part
	   of the LIA solvers
	 * infeasible subset, and indicates if no active domain could be
	   bloated, because the
	 * maximal domain of all variables were reached.
	 * @param LIAConflict Infeasible subset of the LIA solver
	 * @return True,  if no active domain was bloated;
	 *       False, otherwise.
	 */
	bool bloatDomains(const FormulaSetT& LIAConflict);

	/**
	 * Analyzes the infeasible subset of the LIA solver and constructs an
	    infeasible
	 * subset of the received constraints. The unsatisfiability cannot be
	    deduced if
	 * the corresponding original constraints contain real valued
	   variables.
	 * @param LIAConflict Infeasible subset of the LIA solver
	 * @return UNSAT,  if an infeasible subset of the received
	   constraints could be constructed;
	 *       UNKNOWN, otherwise.
	 */
	Answer analyzeConflict(const FormulaSetT& LIAConflict);

	/**
	 * Changes the active domain of a variable and adapts its positional
	   notation
	 * to the basis Settings::expansionBase.
	 * @param expansion Expansion data structure thats keeps all needed
	   informations.
	 * @param domain The new domain that shall be active afterwards. Note
	   , that the new
	 *       domain has to contain the currently active interval.
	 */
	void changeActiveDomain(Expansion& expansion, RationalInterval&&
	  domain);

	/**
	 * Asserts/Removes the given formula to/from the LIA solver.
	 * @param formula The formula to assert/remove to the LIA solver.
	 * @param assert True, if formula shall be asserted;
	 *       False, if formula shall be removed.
	 */
	inline void propagateFormula(const FormulaT& formula, bool assert);
  };
}
```

Listing B.4: CSplitModule.cpp

```
/**
 * @file CSplitModule.cpp
 * @author Ömer Sali <oemer.sali@rwth-aachen.de>
 *
 * @version 2018-04-04
 * Created on 2017-11-01.
 */

#include "CSplitModule.h"

namespace smtrat
{
 template<class Settings>
 CSplitModule<Settings>::CSplitModule(const ModuleInput* _formula,
     RuntimeSettings*, Conditionals& _conditionals, Manager* _manager)
   : Module( _formula, _conditionals, _manager )
   , mPurifications()
   , mExpansions()
   , mLinearizations()
   , mVariableBounds()
   , mLIAModel()
   , mCheckedWithBackends(false)
#ifdef SMTRAT_DEVOPTION_Statistics
   , mStatistics(Settings::moduleName)
#endif
 {}

 template<class Settings>
 bool CSplitModule<Settings>::addCore(ModuleInput::const_iterator
     _subformula)
 {
  addReceivedSubformulaToPassedFormula(_subformula);
  const FormulaT& formula{_subformula->formula()};
  if (formula.getType() == carl::FormulaType::FALSE)
   mInfeasibleSubsets.push_back({formula});
  else if (formula.isBound())
  {
   /// Update the variable domain with the asserted bound
   mVariableBounds.addBound(formula, formula);
   const carl::Variable& variable{*formula.variables().begin()};
   auto expansionIter{mExpansions.firstFind(variable)};
   if (expansionIter == mExpansions.end())
    expansionIter = mExpansions.emplace(variable);
   expansionIter->mChangedBounds = true;
   if (mVariableBounds.isConflicting())
    mInfeasibleSubsets.emplace_back(mVariableBounds.getConflict());
  }
  else if (formula.getType() == carl::FormulaType::CONSTRAINT)
  {
   /// Normalize the left hand side of the constraint and turn the
       relation accordingly
   const ConstraintT& constraint{formula.constraint()};
   const Poly normalization{constraint.lhs().normalize()};
   carl::Relation relation{constraint.relation()};
   if (carl::isNegative(constraint.lhs().lcoeff()))
    relation = carl::turnAroundRelation(relation);

   /// Purifiy and discretize the normalized left hand side to construct
        the linearization
   auto linearizationIter{mLinearizations.firstFind(normalization)};
   if (linearizationIter == mLinearizations.end())
   {
    Poly discretization;
    std::vector<Purification *> purifications;
    bool hasRealVariables{false};
    for (TermT term : normalization)
    {
```

```
  if (!term.isConstant())
  {
   size_t realVariables{0};
   for (const auto& exponent : term.monomial()->exponents())
    if (exponent.first.type() == carl::VariableType::VT_REAL)
     realVariables += exponent.second;
   if (realVariables)
   {
    term.coeff() /= carl::pow(Rational(Settings::discrDenom), realVariables);
    hasRealVariables = true;
   }

   if (!term.isLinear())
   {
    Purification& purification{mPurifications[term.monomial()]};
    purifications.emplace_back(&purification);
    term = term.coeff()*purification.mSubstitutions[0];
   }
   else if (realVariables)
   {
    const carl::Variable variable{term.getSingleVariable()};
    auto expansionIter{mExpansions.firstFind(variable)};
    if (expansionIter == mExpansions.end())
     expansionIter = mExpansions.emplace(variable);
    term = term.coeff()*expansionIter->mQuotients[0];
   }
  }
  discretization += term;
 }
 linearizationIter = mLinearizations.emplace(normalization, discretization, std::move(purifications), hasRealVariables);
}
Linearization& linearization{*linearizationIter};
propagateFormula(FormulaT(linearization.mLinearization, linearization.mParity ? carl::turnAroundRelation(relation) : relation), true);
if (linearization.mRelations.empty())
 for (Purification *purification : linearization.mPurifications)
  ++purification->mUsage;
linearization.mRelations.emplace(relation);

/// Check if the asserted relation trivially conflicts with other asserted relations
switch (relation)
{
 case carl::Relation::EQ:
  if (linearization.mRelations.count(carl::Relation::NEQ))
   mInfeasibleSubsets.push_back({
    FormulaT(normalization, carl::Relation::EQ),
    FormulaT(normalization, carl::Relation::NEQ)
   });
  if (linearization.mRelations.count(carl::Relation::LESS))
   mInfeasibleSubsets.push_back({
    FormulaT(normalization, carl::Relation::EQ),
    FormulaT(normalization, carl::Relation::LESS)
   });
  if (linearization.mRelations.count(carl::Relation::GREATER))
   mInfeasibleSubsets.push_back({
    FormulaT(normalization, carl::Relation::EQ),
    FormulaT(normalization, carl::Relation::GREATER)
   });
  break;
 case carl::Relation::NEQ:
  if (linearization.mRelations.count(carl::Relation::EQ))
   mInfeasibleSubsets.push_back({
    FormulaT(normalization, carl::Relation::NEQ),
    FormulaT(normalization, carl::Relation::EQ)
```

```
      });
      break;
     case carl::Relation::LESS:
      if (linearization.mRelations.count(carl::Relation::EQ))
       mInfeasibleSubsets.push_back({
        FormulaT(normalization, carl::Relation::LESS),
        FormulaT(normalization, carl::Relation::EQ)
       });
      if (linearization.mRelations.count(carl::Relation::GEQ))
       mInfeasibleSubsets.push_back({
        FormulaT(normalization, carl::Relation::LESS),
        FormulaT(normalization, carl::Relation::GEQ)
       });
     case carl::Relation::LEQ:
      if (linearization.mRelations.count(carl::Relation::GREATER))
       mInfeasibleSubsets.push_back({
        FormulaT(normalization, relation),
        FormulaT(normalization, carl::Relation::GREATER)
       });
      break;
     case carl::Relation::GREATER:
      if (linearization.mRelations.count(carl::Relation::EQ))
       mInfeasibleSubsets.push_back({
        FormulaT(normalization, carl::Relation::GREATER),
        FormulaT(normalization, carl::Relation::EQ)
       });
      if (linearization.mRelations.count(carl::Relation::LEQ))
       mInfeasibleSubsets.push_back({
        FormulaT(normalization, carl::Relation::GREATER),
        FormulaT(normalization, carl::Relation::LEQ)
       });
     case carl::Relation::GEQ:
      if (linearization.mRelations.count(carl::Relation::LESS))
       mInfeasibleSubsets.push_back({
        FormulaT(normalization, relation),
        FormulaT(normalization, carl::Relation::LESS)
       });
      break;
     default:
      assert(false);
    }
   }
  return mInfeasibleSubsets.empty();
 }

 template<class Settings>
 void CSplitModule<Settings>::removeCore(ModuleInput::const_iterator
     _subformula)
 {
  const FormulaT& formula{_subformula->formula()};
  if (formula.isBound())
  {
   /// Update the variable domain with the removed bound
   mVariableBounds.removeBound(formula, formula);
   mExpansions.firstAt(*formula.variables().begin()).mChangedBounds =
       true;
  }
  else if (formula.getType() == carl::FormulaType::CONSTRAINT)
  {
   /// Normalize the left hand side of the constraint and turn the
       relation accordingly
   const ConstraintT& constraint{formula.constraint()};
   const Poly normalization{constraint.lhs().normalize()};
   carl::Relation relation{constraint.relation()};
   if (carl::isNegative(constraint.lhs().lcoeff()))
    relation = carl::turnAroundRelation(relation);

```

```
    /// Retrieve the normalized constraint and mark the separator object as changed
    Linearization& linearization{mLinearizations.firstAt(normalization)};
    propagateFormula(FormulaT(linearization.mLinearization, linearization.mParity ? carl::turnAroundRelation(relation) : relation), false);
    linearization.mRelations.erase(relation);
    if (linearization.mRelations.empty())
     for (Purification *purification : linearization.mPurifications)
      ++purification->mUsage;
  }
 }

 template<class Settings>
 void CSplitModule<Settings>::updateModel() const
 {
  if(!mModelComputed)
  {
   clearModel();
   if (mCheckedWithBackends)
   {
    getBackendsModel();
    excludeNotReceivedVariablesFromModel();
   }
   else
   {
    for (const Expansion& expansion : mExpansions)
     if (receivedVariable(expansion.mRationalization))
     {
      Rational value{mLIAModel.at(expansion.mDiscretization).asRational()};
      if (expansion.mRationalization.type() == carl::VariableType::VT_REAL)
       value /= Settings::discrDenom;
      mModel.emplace(expansion.mRationalization, value);
     }
   }
   mModelComputed = true;
  }
 }

 template<class Settings>
 Answer CSplitModule<Settings>::checkCore()
 {
  /// Report unsatisfiability if the already found conflicts are still unresolved
  if (!mInfeasibleSubsets.empty())
   return Answer::UNSAT;

  /// Apply the method only if the asserted formula is not trivially undecidable
  if (rReceivedFormula().isConstraintConjunction())
  {
   Answer answer{Answer::UNKNOWN};

   mLIASolver.push();
   if (resetExpansions())
   {
    Answer LIAAnswer{Answer::UNSAT};
    for (size_t i = 1; LIAAnswer == Answer::UNSAT && (!Settings::maxIter || i <= Settings::maxIter); ++i)
    {
     LIAAnswer = mLIASolver.check(true);
     if (LIAAnswer == Answer::SAT)
     {
      mLIAModel = mLIASolver.model();
      answer = Answer::SAT;
     }
```

```
        else if (LIAAnswer == Answer::UNSAT)
        {
          FormulaSetT LIAConflict{mLIASolver.infeasibleSubsets()[0]};
          if (bloatDomains(LIAConflict))
          {
            LIAAnswer = Answer::UNKNOWN;
            answer = analyzeConflict(LIAConflict);
          }
        }
      }
    }
    mLIASolver.pop();

    if (answer != Answer::UNKNOWN)
    {
      mCheckedWithBackends = false;
      return answer;
    }
  }

  /// Check the asserted formula with the backends
  mCheckedWithBackends = true;
  Answer answer{runBackends()};
  if (answer == Answer::UNSAT)
    getInfeasibleSubsets();

  return answer;
}

template<class Settings>
bool CSplitModule<Settings>::resetExpansions()
{
  /// Update the variable domains and watch out for discretization conflicts
  for (Expansion& expansion : mExpansions)
  {
    RationalInterval& maximalDomain{expansion.mMaximalDomain};
    if (expansion.mChangedBounds)
    {
      maximalDomain = mVariableBounds.getInterval(expansion.mRationalization);
      if (expansion.mRationalization.type() == carl::VariableType::VT_REAL)
        maximalDomain *= Rational(Settings::discrDenom);
      maximalDomain.integralPart_assign();
      if (expansion.mMaximalDomain.isUnbounded())
        expansion.mMaximalDomainSize = DomainSize::UNBOUNDED;
      else if (expansion.mMaximalDomain.diameter() > Settings::maxDomainSize)
        expansion.mMaximalDomainSize = DomainSize::LARGE;
      else
        expansion.mMaximalDomainSize = DomainSize::SMALL;
      expansion.mChangedBounds = false;
    }
    if (maximalDomain.isEmpty())
      return false;
    expansion.mActiveDomain = RationalInterval::emptyInterval();
    expansion.mPurifications.clear();
  }

  /// Activate all used purifications bottom-up
  for (auto purificationIter = mPurifications.begin(); purificationIter != mPurifications.end(); ++purificationIter)
  {
    Purification& purification{purificationIter->second};
    if (purification.mUsage)
    {
      carl::Monomial::Arg monomial{purificationIter->first};
```

```

    /// Find set of variables with maximal domain size
    carl::Variables maxVariables;
    DomainSize maxDomainSize{DomainSize::SMALL};
    for (const auto& exponent : monomial->exponents())
    {
     const carl::Variable& variable{exponent.first};
     auto expansionIter{mExpansions.firstFind(variable)};
     if (expansionIter == mExpansions.end())
      expansionIter = mExpansions.emplace(variable);
     Expansion& expansion{*expansionIter};

     if (maxDomainSize <= expansion.mMaximalDomainSize)
     {
      if (maxDomainSize < expansion.mMaximalDomainSize)
      {
       maxVariables.clear();
       maxDomainSize = expansion.mMaximalDomainSize;
      }
      maxVariables.emplace(variable);
     }
    }

    /// Find a locally optimal reduction for the monomial
    const auto isReducible = [&](const auto& purificationsEntry) {
     return purificationsEntry.second.mActive
      && monomial->divisible(purificationsEntry.first)
      && std::any_of(
       maxVariables.begin(),
       maxVariables.end(),
       [&](const carl::Variable& variable) {
        return purificationsEntry.first->has(variable);
       }
      );
    };
    auto reductionIter{std::find_if(std::make_reverse_iterator(
       purificationIter), mPurifications.rend(), isReducible)};

    /// Activate the sequence of reductions top-down
    carl::Monomial::Arg guidance;
    if (reductionIter == mPurifications.rend())
     monomial->divide(*maxVariables.begin(), guidance);
    else
     monomial->divide(reductionIter->first, guidance);
    auto hintIter{purificationIter};
    for (const auto& exponentPair : guidance->exponents())
    {
     const carl::Variable& variable{exponentPair.first};
     Expansion& expansion{mExpansions.firstAt(variable)};
     for (carl::exponent exponent = 1; exponent <= exponentPair.second;
        ++exponent)
     {
      hintIter->second.mActive = true;
      expansion.mPurifications.emplace_back(&hintIter->second);
      monomial->divide(variable, monomial);
      if (monomial->isAtMostLinear())
       hintIter->second.mReduction = mExpansions.firstAt(monomial->
          getSingleVariable()).mQuotients[0];
      else
      {
       auto temp{mPurifications.emplace_hint(hintIter, std::
          piecewise_construct, std::make_tuple(monomial), std::
          make_tuple())};
       hintIter->second.mReduction = temp->second.mSubstitutions[0];
       hintIter = temp;
      }
     }
```

```
      }
     }
     else
      purification.mActive = false;
    }

    /// Activate expansions that are used for case splits and deactivate
       them otherwise
    for (Expansion& expansion : mExpansions)
    {
     /// Calculate the center point where the initial domain is located
     expansion.mNucleus = ZERO_RATIONAL;
     if (expansion.mMaximalDomain.lowerBoundType() != carl::BoundType::
        INFTY
      && expansion.mNucleus < expansion.mMaximalDomain.lower())
      expansion.mNucleus = expansion.mMaximalDomain.lower();
     else if (expansion.mMaximalDomain.upperBoundType() != carl::BoundType
        ::INFTY
      && expansion.mNucleus > expansion.mMaximalDomain.upper())
      expansion.mNucleus = expansion.mMaximalDomain.upper();

     /// Calculate and activate the corresponding domain
     RationalInterval domain(0, 1);
     domain.mul_assign(Rational(Settings::initialRadius));
     domain.add_assign(expansion.mNucleus);
     domain.intersect_assign(expansion.mMaximalDomain);
     changeActiveDomain(expansion, std::move(domain));
    }

   return true;
  }

  template<class Settings>
  bool CSplitModule<Settings>::bloatDomains(const FormulaSetT&
      LIAConflict)
  {
   /// Data structure for potential bloating candidates
   struct Candidate
   {
    Expansion& mExpansion;
    const Rational mDirection;
    const Rational mRadius;

    Candidate(Expansion& expansion, Rational&& direction, Rational&&
       radius)
     : mExpansion(expansion)
     , mDirection(std::move(direction))
     , mRadius(std::move(radius))
    {}

    bool operator<(const Candidate& rhs) const
    {
     if (mDirection*rhs.mDirection == ONE_RATIONAL)
      return mRadius < rhs.mRadius;
     else if (mDirection == ONE_RATIONAL)
      return mRadius < Rational(Settings::thresholdRadius);
     else
      return rhs.mRadius >= Rational(Settings::thresholdRadius);
    }
   };
   std::set<Candidate> candidates;

   /// Scan the infeasible subset of the LIA solver for potential
      candidates
   for (const FormulaT& formula : LIAConflict)
    if (formula.isBound())
    {
     const ConstraintT& constraint{formula.constraint()};
```

```
    const carl::Variable& variable{*constraint.variables().begin()};
    auto expansionIter{mExpansions.secondFind(variable)};
    if (expansionIter != mExpansions.end())
    {
     Expansion& expansion{*expansionIter};
     Rational direction;
     if (constraint.isLowerBound()
      && (expansion.mMaximalDomain.lowerBoundType() == carl::BoundType::INFTY
       || expansion.mMaximalDomain.lower() < expansion.mActiveDomain.lower()))
      direction = MINUS_ONE_RATIONAL;
     else if (constraint.isUpperBound()
      && (expansion.mMaximalDomain.upperBoundType() == carl::BoundType::INFTY
       || expansion.mMaximalDomain.upper() > expansion.mActiveDomain.upper()))
      direction  = ONE_RATIONAL;
     if (direction != ZERO_RATIONAL)
     {
      Rational radius{(direction*(expansion.mActiveDomain-expansion.mNucleus)).upper()};

      if (!Settings::maximalRadius
       || radius <= Settings::maximalRadius)
      {
       candidates.emplace(expansion, std::move(direction), std::move(radius));
       if (Settings::maxBloatedDomains
        && candidates.size() > Settings::maxBloatedDomains)
        candidates.erase(std::prev(candidates.end()));
      }
     }
    }
   }

  /// Change the active domain of the candidates with highest priority
  for (const Candidate& candidate : candidates)
  {
   RationalInterval domain;
   if (candidate.mRadius <= Settings::thresholdRadius)
    domain = RationalInterval(0, ONE_RATIONAL);
   else if (candidate.mExpansion.mPurifications.empty())
    domain = RationalInterval(0, carl::BoundType::WEAK, 0, carl::BoundType::INFTY);
   else
    domain = RationalInterval(0, candidate.mRadius);
   domain.mul_assign(candidate.mDirection);
   domain.add_assign(candidate.mExpansion.mActiveDomain);
   domain.intersect_assign(candidate.mExpansion.mMaximalDomain);
   changeActiveDomain(candidate.mExpansion, std::move(domain));
  }

  /// Report if any variable domain was bloated
  return candidates.empty();
 }

 template<class Settings>
 Answer CSplitModule<Settings>::analyzeConflict(const FormulaSetT& LIAConflict)
 {
  /// Construct an infeasible subset from the LIA conflict
  FormulaSetT conflict;
  for (const FormulaT& formula : LIAConflict)
  {
   if (formula.isBound())
   {
```

```
      auto expansionIter{mExpansions.secondFind(*formula.variables().begin
        ())};
      if (expansionIter != mExpansions.end())
      {
        const Expansion& expansion{*expansionIter};
        if (expansion.mRationalization.type() == carl::VariableType::
          VT_REAL
          || expansion.mMaximalDomain != expansion.mActiveDomain)
          return Answer::UNKNOWN;
        else
        {
          FormulaSetT boundOrigins{mVariableBounds.getOriginSetOfBounds(
            expansion.mRationalization)};
          conflict.insert(boundOrigins.begin(), boundOrigins.end());
        }
      }
    }
    else if (formula.getType() == carl::FormulaType::CONSTRAINT)
    {
      const ConstraintT& constraint{formula.constraint()};
      auto linearizationIter{mLinearizations.secondFind(constraint.lhs().
        normalize())};
      if (linearizationIter != mLinearizations.end())
      {
        const Linearization& linearization{*linearizationIter};
        if (linearization.mHasRealVariables)
          return Answer::UNKNOWN;
        else
        {
          carl::Relation relation{constraint.relation()};
          if (carl::isNegative(constraint.lhs().lcoeff()) != linearization.
            mParity)
            relation = carl::turnAroundRelation(relation);
          conflict.emplace(linearization.mNormalization, relation);
        }
      }
    }
  }

  mInfeasibleSubsets.emplace_back(std::move(conflict));
  return Answer::UNSAT;
}

template<class Settings>
void CSplitModule<Settings>::changeActiveDomain(Expansion& expansion,
    RationalInterval&& domain)
{
  RationalInterval activeDomain{move(expansion.mActiveDomain)};
  expansion.mActiveDomain = domain;

  /// Update the variable bounds
  if (!activeDomain.isEmpty())
  {
    if (activeDomain.lowerBoundType() != carl::BoundType::INFTY
      && (domain.lowerBoundType() == carl::BoundType::INFTY
        || domain.lower() != activeDomain.lower()
        || domain.isEmpty()))
      propagateFormula(FormulaT(expansion.mQuotients[0]-Poly(activeDomain.
        lower()), carl::Relation::GEQ), false);
    if (activeDomain.upperBoundType() != carl::BoundType::INFTY
      && (domain.upperBoundType() == carl::BoundType::INFTY
        || domain.upper() != activeDomain.upper()
        || domain.isEmpty()))
      propagateFormula(FormulaT(expansion.mQuotients[0]-Poly(activeDomain.
        upper()), carl::Relation::LEQ), false);
  }
  if (!domain.isEmpty())
```

```
  {
   if (domain.lowerBoundType() != carl::BoundType::INFTY
    && (activeDomain.lowerBoundType() == carl::BoundType::INFTY
     || activeDomain.lower() != domain.lower()
     || activeDomain.isEmpty()))
    propagateFormula(FormulaT(expansion.mQuotients[0]-Poly(domain.lower()), carl::Relation::GEQ), true);
   if (domain.upperBoundType() != carl::BoundType::INFTY
    && (activeDomain.upperBoundType() == carl::BoundType::INFTY
     || activeDomain.upper() != domain.upper()
     || activeDomain.isEmpty()))
    propagateFormula(FormulaT(expansion.mQuotients[0]-Poly(domain.upper()), carl::Relation::LEQ), true);
  }

  /// Check if the digits of the expansion need to be encoded
  if (expansion.mPurifications.empty())
  {
   activeDomain = RationalInterval::emptyInterval();
   domain = RationalInterval::emptyInterval();
  }

  /// Update the case splits of the corresponding digits
  for (size_t i = 0; activeDomain != domain; ++i)
  {
   if (domain.diameter() <= Settings::maxDomainSize)
   {
    /// Update the currently active linear encoding
    Rational lower{activeDomain.isEmpty() ? domain.lower() : activeDomain.lower()};
    Rational upper{activeDomain.isEmpty() ? domain.lower() : activeDomain.upper()+ONE_RATIONAL};
    for (const Purification *purification : expansion.mPurifications)
    {
     for (Rational alpha = domain.lower(); alpha < lower; ++alpha)
      propagateFormula(
       FormulaT(
        carl::FormulaType::IMPLIES,
        FormulaT(Poly(expansion.mQuotients[i])-Poly(alpha), carl::Relation::EQ),
        FormulaT(Poly(purification->mSubstitutions[i])-Poly(alpha)*Poly(purification->mReduction), carl::Relation::EQ)
       ),
       true
      );
     for (Rational alpha = upper; alpha <= domain.upper(); ++alpha)
      propagateFormula(
       FormulaT(
        carl::FormulaType::IMPLIES,
        FormulaT(Poly(expansion.mQuotients[i])-Poly(alpha), carl::Relation::EQ),
        FormulaT(Poly(purification->mSubstitutions[i])-Poly(alpha)*Poly(purification->mReduction), carl::Relation::EQ)
       ),
       true
      );
    }
   }
   else if (activeDomain.diameter() <= Settings::maxDomainSize)
   {
    /// Switch from the linear to a logarithmic encoding
    if (expansion.mQuotients.size() <= i+1)
    {
     expansion.mQuotients.emplace_back(carl::freshIntegerVariable());
     expansion.mRemainders.emplace_back(carl::freshIntegerVariable());
    }
    for (Purification *purification : expansion.mPurifications)
```

```
    {
     if (purification->mSubstitutions.size() <= i+1)
      purification->mSubstitutions.emplace_back(carl::freshIntegerVariable());
     for (Rational alpha = activeDomain.lower(); alpha <= activeDomain.upper(); ++alpha)
      propagateFormula(
       FormulaT(
        carl::FormulaType::IMPLIES,
        FormulaT(Poly(expansion.mQuotients[i])-Poly(alpha), carl::Relation::EQ),
        FormulaT(Poly(purification->mSubstitutions[i])-Poly(alpha)*Poly(purification->mReduction), carl::Relation::EQ)
       ),
       false
      );
     for (Rational alpha = ZERO_RATIONAL; alpha < Settings::expansionBase; ++alpha)
      propagateFormula(
       FormulaT(
        carl::FormulaType::IMPLIES,
        FormulaT(Poly(expansion.mRemainders[i])-Poly(alpha), carl::Relation::EQ),
        FormulaT(Poly(purification->mSubstitutions[i])-Poly(Settings::expansionBase)*Poly(purification->mSubstitutions[i+1])-Poly(alpha)*Poly(purification->mReduction), carl::Relation::EQ)
       ),
       true
      );
    }
    propagateFormula(FormulaT(Poly(expansion.mQuotients[i])-Poly(Settings::expansionBase)*Poly(expansion.mQuotients[i+1])-Poly(expansion.mRemainders[i]), carl::Relation::EQ), true);
    propagateFormula(FormulaT(Poly(expansion.mRemainders[i]), carl::Relation::GEQ), true);
    propagateFormula(FormulaT(Poly(expansion.mRemainders[i])-Poly(Settings::expansionBase-1), carl::Relation::LEQ), true);
   }

   /// Calculate the domain of the next digit
   if (!activeDomain.isEmpty())
    if (activeDomain.diameter() <= Settings::maxDomainSize)
     activeDomain = RationalInterval::emptyInterval();
    else
     activeDomain = carl::floor(activeDomain/Rational(Settings::expansionBase));
   if (!domain.isEmpty())
    if (domain.diameter() <= Settings::maxDomainSize)
     domain = RationalInterval::emptyInterval();
    else
     domain = carl::floor(domain/Rational(Settings::expansionBase));

   /// Update the variable bounds of the next digit
   if (!activeDomain.isEmpty())
   {
    if (domain.isEmpty() || domain.lower() != activeDomain.lower())
     propagateFormula(FormulaT(expansion.mQuotients[i+1]-Poly(activeDomain.lower()), carl::Relation::GEQ), false);
    if (domain.isEmpty() || domain.upper() != activeDomain.upper())
     propagateFormula(FormulaT(expansion.mQuotients[i+1]-Poly(activeDomain.upper()), carl::Relation::LEQ), false);
   }
   if (!domain.isEmpty())
   {
    if (activeDomain.isEmpty() || activeDomain.lower() != domain.lower())
     propagateFormula(FormulaT(expansion.mQuotients[i+1]-Poly(domain.
```

```
                lower()), carl::Relation::GEQ), true);
        if (activeDomain.isEmpty() || activeDomain.upper() != domain.upper()
            )
          propagateFormula(FormulaT(expansion.mQuotients[i+1]-Poly(domain.
                upper()), carl::Relation::LEQ), true);
      }
    }
  }

  template<class Settings>
  inline void CSplitModule<Settings>::propagateFormula(const FormulaT&
        formula, bool assert)
  {
    if (assert)
      mLIASolver.add(formula);
    else
      mLIASolver.remove(std::find(mLIASolver.formulaBegin(), mLIASolver.
            formulaEnd(), formula));
  }
}

#include "Instantiation.h"
```

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